■ Based On Derivatives

VERY SHORT ANSWER TYPE QUESTIONS

Q01. Given $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$ $\Rightarrow y = \cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$

$\Rightarrow y = \frac{\pi}{2}$ $\therefore \frac{dy}{dx} = 0$

Q02. We have $f(x) = \ln x \quad \Rightarrow f(\ln x) = \ln(\ln x)$

$\therefore$ differential coefficient of $f(\ln x) = \frac{d}{dx}[\ln(\ln x)] = \frac{1}{\ln x} \times \frac{1}{x} = (x \ln x)^{-1}$.

Q03. We have $y = \log_{\sqrt{x}} x \quad \Rightarrow y = \frac{\log_e x}{\log_e \sqrt{x}} = \frac{\log_e x}{\frac{1}{2}\log_e x} = 2$ $\quad \therefore \frac{dy}{dx} = 0$.

Q04. Given $y = \tan^{-1}\left(\frac{1-x}{1+x}\right) + \tan^{-1}\left(\frac{x+2}{1-2x}\right)$ $\Rightarrow y = \tan^{-1}1 - \tan^{-1}x + \tan^{-1}x + \tan^{-1}2$

$\Rightarrow y = \tan^{-1}1 + \tan^{-1}2$ $\quad \therefore \frac{dy}{dx} = 0$

Hence it is clear that the first order derivative of y vanishes.

Q05. It is true only for positive values of x because for $\log x$ to be defined, x must be positive real no.

Q06. We have $f(x) = \sqrt{x \log_e x}$ $\quad \Rightarrow f'(x) = \frac{1}{2\sqrt{x \log_e x}}\left[x \times \frac{1}{x} + \log_e x\right]$

$\therefore f'(e) = \frac{1}{2\sqrt{e \log_e e}}[1 + \log_e e] = \frac{1}{2\sqrt{e}}[1+1] = \frac{1}{\sqrt{e}}$.

Q07. Given $y = 2\sin x^\circ + \frac{3}{2}\cos x^\circ$ $\quad \Rightarrow y = 2\sin\frac{\pi x}{180} + \frac{3}{2}\cos\frac{\pi x}{180}$

$\therefore \frac{dy}{dx} = 2 \times \frac{\pi}{180}\cos\frac{\pi x}{180} - \frac{3}{2} \times \frac{\pi}{180}\sin\frac{\pi x}{180}$ $\quad \Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\left[2\cos x^\circ - \frac{3}{2}\sin x^\circ\right]$

Q08. Let $y = \sin x$ and, $z = \cos x$ $\quad \Rightarrow \frac{dy}{dx} = \cos x, \ \frac{dz}{dx} = -\sin x$

$\therefore \frac{dy}{dz} = \frac{dy}{dx} \times \frac{dx}{dz} = \cos x \times \frac{1}{-\sin x} = -\cot x$

Q09. We have $f(x) = |x| = \sqrt{x^2}$ $\quad \Rightarrow f'(x) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}, x \neq 0$.

Q10. We have $f(x) = x\,g(x)$ $\quad \Rightarrow f'(x) = x\,g'(x) + g(x)$

$\therefore f'(0) = 0 \times g'(0) + g(0) = 0 + 3 = 3$

Q11. Given $y = \sin\left(\frac{\pi}{2} - \cos^{-1}x\right)$ $\quad \Rightarrow y = \cos\left(\cos^{-1}x\right)$ $\quad \Rightarrow y = x$ $\quad \therefore \frac{dy}{dx} = 1$

Q12. We have $y = (\ln \ln x)^2$ $\quad \therefore \frac{dy}{dx} = 2(\ln \ln x) \times \frac{1}{\ln x} \times \frac{1}{x} = \frac{2\ln(\ln x)}{x \ln x}$

Q13. We have $y = \log_{\sqrt{e}} \sin x$ $\quad \Rightarrow y = \frac{\log_e \sin x}{\log_e \sqrt{e}} = \frac{\log_e \sin x}{\frac{1}{2}\log_e e} = 2\log_e \sin x$

$\therefore \frac{dy}{dx} = 2\frac{\cos x}{\sin x} = 2\cot x$

Q14. Given $y = \log_x \sqrt{x} = \frac{\log_e \sqrt{x}}{\log_e x} = \frac{1}{2}$ $\quad \therefore \frac{dy}{dx} = 0$

Q15. We have $y = \sin(\sin^{-1} x + \cos^{-1} x)$ $\Rightarrow y = \sin\frac{\pi}{2} = 1$ $\quad \therefore D(y) = 0$

Q16. Let $y = \log_5(\log x)$ $\Rightarrow y = \frac{\log_e(\log x)}{\log_e 5}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\log_e 5} \times \frac{1}{\log x} \times \frac{1}{x} = \frac{\log_5 e}{x \log x}$.

Q17. Let $y = 10^{5\log_{10} x}$ $\Rightarrow y = 10^{\log_{10} x^5} = x^5$ $\Rightarrow \frac{dy}{dx} = 5x^4$.

Q18. Let $y = \log_{x^7} x = \frac{\log_e x}{\log_e x^7} = \frac{1}{7}$ $\quad \therefore \frac{dy}{dx} = 0$.

Q19. Let $y = \log_x x = 1$ $\quad \therefore \frac{dy}{dx} = 0$.

Q20. Let $y = \sqrt{\sin^{-1}\sqrt{x}}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin^{-1}\sqrt{x}}} \times \frac{d}{dx}[\sin^{-1}\sqrt{x}]$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin^{-1}\sqrt{x}}} \times \frac{1}{\sqrt{1-x}} \times \frac{d}{dx}\sqrt{x}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin^{-1}\sqrt{x}}} \times \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{1}{4\sqrt{x(1-x)\sin^{-1}\sqrt{x}}}$

Q21. Let $y = \log_{10} x = \frac{\log_e x}{\log_e 10}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\log_e 10} \times \frac{1}{x}$ $\quad \therefore \frac{dy}{dx} = \frac{\log_{10} e}{x}$.

Q22. Let $y = f(e^{\tan x})$ $\Rightarrow \frac{dy}{dx} = f'(e^{\tan x}) \times \frac{d}{dx}(e^{\tan x})$

$\Rightarrow \frac{dy}{dx} = f'(e^{\tan x}) \times e^{\tan x}.\sec^2 x$ $\Rightarrow \left.\frac{dy}{dx}\right]_{at\ x=0} = f'(e^{\tan 0}) \times e^{\tan 0} \times \sec^2 0$

$\Rightarrow \left.\frac{dy}{dx}\right]_{at\ x=0} = f'(1) \times e^0 \times 1$ $\quad \therefore \left.\frac{dy}{dx}\right]_{at\ x=0} = 5$

Q23. Here $e^y(x+1) = 1 \Rightarrow e^y + (x+1)e^y\frac{dy}{dx} = 0 \Rightarrow e^y + 1 \times \frac{dy}{dx} = 0$ $\therefore \frac{dy}{dx} = -e^y$.

◘ Based On Recapitulation Of Derivatives

NOTE that Q01 to Q16 in this section have been left as an exercise (for recapitulation) for you.

Q17. Given $y = \sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}$ $\Rightarrow y^2 = \frac{x}{a} + \frac{a}{x} + 2$ $\Rightarrow 2y \times \frac{dy}{dx} = \frac{1}{a} - \frac{a}{x^2}$

Multiplying by x both sides, $2xy\frac{dy}{dx} = \frac{x}{a} - \frac{a}{x}$ **H. P.**

SHORT ANSWER TYPE QUESTIONS

◘ Based On Derivatives Of Inverse Trigonometric Functions

Q01. Let $y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$ Put $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a} \ldots(i)$

$\Rightarrow y=\tan^{-1}\left(\frac{3a^2(a\tan\theta)-a^3\tan^3\theta}{a^3-3a(a^2\tan^2\theta)}\right)$ $\quad\Rightarrow y=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$\Rightarrow y=\tan^{-1}(\tan 3\theta)=3\theta$ $\quad$ By (i), $y=3\tan^{-1}\frac{x}{a}$

$\Rightarrow \frac{dy}{dx}=3\left[\frac{1}{1+\left(\frac{x}{a}\right)^2}\right]\times\frac{1}{a}$ $\quad\therefore \frac{dy}{dx}=\frac{3a}{a^2+x^2}$

Q02. Let $y=\sin^{-1}\left(x\sqrt{1-x^4}+x^2\sqrt{1-x^2}\right)$ $\quad\Rightarrow y=\sin^{-1}\left(x\sqrt{1-(x^2)^2}+x^2\sqrt{1-x^2}\right)$

Put $x=\sin\theta,\ x^2=\sin\beta\Rightarrow\theta=\sin^{-1}x,\ \beta=\sin^{-1}x^2\ \ldots$(i)

$\therefore y=\sin^{-1}\left(\sin\theta\sqrt{1-\sin^2\beta}+\sin\beta\sqrt{1-\sin^2\theta}\right)$ $\quad\Rightarrow y=\sin^{-1}(\sin\theta\cos\beta+\sin\beta\cos\theta)$

$\Rightarrow y=\sin^{-1}\sin(\theta+\beta)=\theta+\beta$ $\quad$ By (i), $y=\sin^{-1}x+\sin^{-1}x^2$

On differentiating w.r.t. x both sides, $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^4}}\times\frac{d}{dx}(x^2)$

$\therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{2x}{\sqrt{1-x^4}}$.

Q03. Let $y=\cos^{-1}\left(x\sqrt{1-x}-\sqrt{x-x^3}\right)$ $\quad\Rightarrow y=\cos^{-1}\left(x\sqrt{1-[\sqrt{x}]^2}-\sqrt{x}\sqrt{1-x^2}\right)$

Put $x=\cos\theta,\ \sqrt{x}=\cos\beta\Rightarrow\theta=\cos^{-1}x,\ \beta=\cos^{-1}\sqrt{x}\ \ldots$(i)

$\Rightarrow y=\cos^{-1}\left(\cos\theta\sqrt{1-\cos^2\beta}-\cos\beta\sqrt{1-\cos^2\theta}\right)$ $\quad\Rightarrow y=\cos^{-1}(\cos\theta\sin\beta-\cos\beta\sin\theta)$

$\Rightarrow y=\cos^{-1}\sin(\beta-\theta)$ $\quad\Rightarrow y=\cos^{-1}\cos\left[\frac{\pi}{2}-(\beta-\theta)\right]=\frac{\pi}{2}-(\beta-\theta)$

$\Rightarrow y=\frac{\pi}{2}-\cos^{-1}\sqrt{x}+\cos^{-1}x$ $\quad\Rightarrow y=\frac{\pi}{2}-\cos^{-1}\sqrt{x}+\cos^{-1}x$

$\Rightarrow \frac{dy}{dx}=0-\left(-\frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}}\right)-\frac{1}{\sqrt{1-x^2}}$ $\quad\therefore \frac{dy}{dx}=\frac{1}{2\sqrt{x-x^2}}-\frac{1}{\sqrt{1-x^2}}$

Q04. Let $y=\tan^{-1}\left(\frac{5ax}{a^2-6x^2}\right)$ $\quad\Rightarrow y=\tan^{-1}\left(\frac{\frac{5ax}{a^2}}{\frac{a^2}{a^2}-\frac{6x^2}{a^2}}\right)$ $\quad\Rightarrow y=\tan^{-1}\left(\frac{\frac{3x}{a}+\frac{2x}{a}}{1-\frac{3x}{a}\times\frac{2x}{a}}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{3x}{a}\right)+\tan^{-1}\left(\frac{2x}{a}\right)$ $\quad\Rightarrow \frac{dy}{dx}=\frac{1}{1+\frac{9x^2}{a^2}}\times\frac{3}{a}+\frac{1}{1+\frac{4x^2}{a^2}}\times\frac{2}{a}$

$\Rightarrow \frac{dy}{dx}=\frac{3a}{a^2+9x^2}+\frac{2a}{a^2+4x^2}$ $\quad\therefore \frac{dy}{dx}=\frac{5a(a^2+6x^2)}{(a^2+9x^2)(a^2+4x^2)}$

Another similar sum : If $y=\tan^{-1}\frac{5x}{1-6x^2},\ -\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$, then prove that

$\frac{dy}{dx}=\frac{2}{1+4x^2}+\frac{3}{1+9x^2}$.

Sol. Here $y=\tan^{-1}\frac{5x}{1-6x^2}=\tan^{-1}\left(\frac{3x+2x}{1-3x.2x}\right)=\tan^{-1}3x+\tan^{-1}2x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{d}{dx}\left(\tan^{-1}3x + \tan^{-1}2x\right)$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{3}{1+9x^2} + \dfrac{2}{1+4x^2}$.

Q05. Let $y = \tan^{-1}\left(\dfrac{x}{1+6x^2}\right)$ $\quad \Rightarrow y = \tan^{-1}\left(\dfrac{3x-2x}{1+3x.2x}\right) = \tan^{-1}3x - \tan^{-1}2x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3}{1+9x^2} - \dfrac{2}{1+4x^2}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{\left(1-6x^2\right)}{\left(1+9x^2\right)\left(1+4x^2\right)}$

Q06. Let $y = \tan^{-1}\left(\dfrac{\sqrt{x}-x}{1+x^{3/2}}\right)$ $\quad \Rightarrow y = \tan^{-1}\left(\dfrac{\sqrt{x}-x}{1+\sqrt{x}.x}\right)$ $\quad \Rightarrow y = \tan^{-1}\sqrt{x} - \tan^{-1}x$

$\Rightarrow y = \tan^{-1}\sqrt{x} - \tan^{-1}x$ $\quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1+x} \times \dfrac{1}{2\sqrt{x}} - \dfrac{1}{1+x^2}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}\,(1+x)} - \dfrac{1}{1+x^2}$

Q07. Let $y = \sin^{-1}\left(\dfrac{3x+4\sqrt{1-x^2}}{5}\right)$ $\quad \Rightarrow y = \sin^{-1}\left(\dfrac{3}{5}x + \dfrac{4}{5}\sqrt{1-x^2}\right)$

Put $\dfrac{3}{5} = \cos\theta \Rightarrow \dfrac{4}{5} = \sin\theta$ and, $x = \sin\beta \;\Rightarrow \sqrt{1-x^2} = \cos\beta$

$\Rightarrow \theta = \cos^{-1}\dfrac{3}{5},\; \beta = \sin^{-1}x \ldots$(i)

$\Rightarrow y = \sin^{-1}\left(\cos\theta\sin\beta + \sin\theta\cos\beta\right)$ $\quad \Rightarrow y = \sin^{-1}\sin\left(\beta+\theta\right)$ $\quad \Rightarrow y = \theta + \beta$

By using (i), $y = \cos^{-1}\dfrac{3}{5} + \sin^{-1}x$ $\quad \Rightarrow \dfrac{dy}{dx} = 0 + \dfrac{1}{\sqrt{1-x^2}}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

Another similar sum : Find $\dfrac{dy}{dx}$ if $y = \sin^{-1}\left[\dfrac{6x-4\sqrt{1-4x^2}}{5}\right]$.

Sol. We have $y = \sin^{-1}\left[\dfrac{6x-4\sqrt{1-4x^2}}{5}\right]$ $\quad$ Put $2x = \sin\theta \Rightarrow \theta = \sin^{-1}2x$...(i)

$\therefore y = \sin^{-1}\left[\dfrac{3\sin\theta - 4\sqrt{1-\sin^2\theta}}{5}\right]$ $\quad \Rightarrow y = \sin^{-1}\left[\dfrac{3}{5}\sin\theta - \dfrac{4}{5}\cos\theta\right]$

Let $\dfrac{3}{5} = \cos\alpha \Rightarrow \sin\alpha = \dfrac{4}{5} \ldots$(ii) $\quad \therefore y = \sin^{-1}\left[\cos\alpha\sin\theta - \sin\alpha\cos\theta\right]$

$\Rightarrow y = \sin^{-1}\left[\sin(\theta-\alpha)\right] = \theta - \alpha$ $\quad \Rightarrow y = \sin^{-1}2x - \sin^{-1}\dfrac{4}{5}$ $\quad$ [By (i) and (ii)

$\therefore \dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-4x^2}} - 0 = \dfrac{2}{\sqrt{1-4x^2}}$.

Q08. Let $y = \sin^{-1}\left(\dfrac{\sqrt{1-x}+\sqrt{1+x}}{2}\right)$ $\quad$ Put $x = \cos2\theta \Rightarrow \theta = \dfrac{1}{2}\cos^{-1}x \ldots$(i)

$\Rightarrow y = \sin^{-1}\left(\dfrac{\sqrt{1-\cos2\theta}+\sqrt{1+\cos2\theta}}{2}\right)$ $\quad \Rightarrow y = \sin^{-1}\left(\dfrac{\sqrt{2}\,|\sin\theta| + \sqrt{2}\,|\cos\theta|}{2}\right)$

$\left[\begin{array}{l} \because 0 < x < 1 \Rightarrow 0 < \cos2\theta < 1 \\ \Rightarrow 0 < 2\theta < \dfrac{\pi}{2} \Rightarrow 0 < \theta < \dfrac{\pi}{4} \end{array}\right]$ $\quad \therefore y = \sin^{-1}\left(\dfrac{\sqrt{2}\sin\theta + \sqrt{2}\cos\theta}{2}\right)$

$\Rightarrow y=\sin^{-1}\left(\frac{1}{\sqrt{2}}\times\sin\theta+\frac{1}{\sqrt{2}}\times\cos\theta\right)$ $\Rightarrow y=\sin^{-1}\left(\cos\frac{\pi}{4}\sin\theta+\sin\frac{\pi}{4}\cos\theta\right)$

$\Rightarrow y=\sin^{-1}\sin\left(\theta+\frac{\pi}{4}\right)=\theta+\frac{\pi}{4}$ By (i), $y=\frac{1}{2}\cos^{-1}x+\frac{\pi}{4}$ $\therefore\frac{dy}{dx}=-\frac{1}{2\sqrt{1-x^2}}$

Q09. Let $y=\sin^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{2}\right)$ Put $x=\cos2\theta\Rightarrow\theta=\frac{1}{2}\cos^{-1}x$ ...(i)

$\Rightarrow y=\sin^{-1}\left(\frac{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}{2}\right)$ $\Rightarrow y=\sin^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{2}\right)$

$\Rightarrow y=\sin^{-1}\left(\frac{1}{\sqrt{2}}\times\cos\theta-\frac{1}{\sqrt{2}}\times\sin\theta\right)$ $\Rightarrow y=\sin^{-1}\left(\sin\frac{\pi}{4}\cos\theta-\cos\frac{\pi}{4}\sin\theta\right)$

$\Rightarrow y=\sin^{-1}\sin\left(\frac{\pi}{4}-\theta\right)=\frac{\pi}{4}-\theta$ By (i), $y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$ $\therefore\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$

Q10. Let $y=\sin^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$ $\Rightarrow y=\sin^{-1}\left(\frac{x-\frac{1}{x}}{x+\frac{1}{x}}\right)=\sin^{-1}\left(\frac{x^2-1}{x^2+1}\right)=\sin^{-1}\left(\frac{-(1-x^2)}{x^2+1}\right)$

$\Rightarrow y=-\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ Put $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$ ...(i)

$\Rightarrow y=-\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$ $\Rightarrow y=-\sin^{-1}(\cos2\theta)=-\sin^{-1}\sin\left(\frac{\pi}{2}-2\theta\right)=-\left(\frac{\pi}{2}-2\theta\right)$

By using (i), $y=-\frac{\pi}{2}+2\tan^{-1}x$ $\therefore\frac{dy}{dx}=\frac{2}{1+x^2}$

Q11. Let $y=\cot^{-1}\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}$ Put $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$ ...(i)

$\Rightarrow y=\cot^{-1}\sqrt{\frac{\sqrt{1+\tan^2\theta}+\tan\theta}{\sqrt{1+\tan^2\theta}-\tan\theta}}$ $\Rightarrow y=\cot^{-1}\sqrt{\frac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta}}=\cot^{-1}\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$

$\Rightarrow y=\cot^{-1}\sqrt{\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}$ $\Rightarrow y=\cot^{-1}\sqrt{\frac{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}}$

$\Rightarrow y=\cot^{-1}\left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)$ $\Rightarrow y=\cot^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)=\cot^{-1}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$\Rightarrow y=\frac{\pi}{2}-\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ $\Rightarrow y=\frac{\pi}{4}-\frac{\theta}{2}$

By (i), $y=\frac{\pi}{4}-\frac{1}{2}\tan^{-1}x$ $\therefore\frac{dy}{dx}=-\frac{1}{2(1+x^2)}$.

Q12. Let $y=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$ Put $x=\cos2\theta\Rightarrow\theta=\frac{1}{2}\cos^{-1}x$ ...(i)

$\Rightarrow y=\tan^{-1}\left(\dfrac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$ $\Rightarrow y=\tan^{-1}\left(\dfrac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$

$\Rightarrow y=\tan^{-1}\left(\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$ $\Rightarrow y=\tan^{-1}\left(\dfrac{1-\tan\theta}{1+\tan\theta}\right)$ $\Rightarrow y=\tan^{-1}\tan\left(\dfrac{\pi}{4}-\theta\right)$

$\Rightarrow y=\dfrac{\pi}{4}-\theta=\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x$ [By (i)

On differentiating w.r.t. x both sides, we get : $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{1-x^2}}$

Q13. Let $y=\cot^{-1}\left(\dfrac{1+x}{1-x}\right)$ Put $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$ …(i)

$\Rightarrow y=\cot^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$ $\Rightarrow y=\cot^{-1}\left(\tan\left(\dfrac{\pi}{4}+\theta\right)\right)$ $\Rightarrow y=\cot^{-1}\cot\left[\dfrac{\pi}{2}-\left(\dfrac{\pi}{4}+\theta\right)\right]$

$\Rightarrow y=\dfrac{\pi}{4}-\theta$ By using (i), $y=\dfrac{\pi}{4}-\tan^{-1}x$ $\therefore\dfrac{dy}{dx}=-\dfrac{1}{1+x^2}$

Q14. Let $y=\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ Put $x=\sin\theta\Rightarrow\theta=\sin^{-1}x$ …(i)

$\Rightarrow y=\tan^{-1}\left(\dfrac{2\sin\theta\sqrt{1-\sin^2\theta}}{1-2\sin^2\theta}\right)$ $\Rightarrow y=\tan^{-1}\left(\dfrac{2\sin\theta\cos\theta}{\cos 2\theta}\right)$ $\Rightarrow y=\tan^{-1}\left(\dfrac{\sin 2\theta}{\cos 2\theta}\right)$

$\Rightarrow y=\tan^{-1}(\tan 2\theta)=2\theta$ $\Rightarrow y=2\sin^{-1}x$ [By using (i)

$\therefore\dfrac{dy}{dx}=\dfrac{2}{\sqrt{1-x^2}}$

Q15. Let $y=\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)=\cos^{-1}(2x^2-1)$ Put $x=\cos\theta\Rightarrow\theta=\cos^{-1}x$ …(i)

$\Rightarrow y=\cos^{-1}(2\cos^2\theta-1)$ $\Rightarrow y=\cos^{-1}(\cos 2\theta)=2\theta$

By using (i), $y=2\cos^{-1}x$ $\therefore\dfrac{dy}{dx}=-\dfrac{2}{\sqrt{1-x^2}}$

Q16. Let $y=\sin^2\left[2\cot^{-1}\sqrt{\dfrac{1+x}{1-x}}\right]$ Put $x=\cos\theta$ …(i)

$\Rightarrow y=\sin^2\left[2\cot^{-1}\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}\right]$ $\Rightarrow y=\sin^2\left[2\cot^{-1}\sqrt{\dfrac{2\cos^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}}\right]$

$\Rightarrow y=\sin^2\left[2\cot^{-1}\cot\dfrac{\theta}{2}\right]=\sin^2\theta=1-\cos^2\theta$

By (i), $y=1-x^2$ $\therefore\dfrac{dy}{dx}=-2x$

Q17. Let $y=\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$ $\Rightarrow y=\sin^{-1}\left(\dfrac{2\times 2^x}{1+(2^x)^2}\right)$ Put $2^x=\tan\theta\Rightarrow\theta=\tan^{-1}2^x$ …(i)

$\Rightarrow y=\sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$ $\Rightarrow y=\sin^{-1}\sin 2\theta$ $\Rightarrow y=2\theta$

By (i), $y = 2\tan^{-1} 2^x$ $\qquad \Rightarrow \dfrac{dy}{dx} = 2 \times \dfrac{1}{1+(2^x)^2} \times \dfrac{d}{dx}[2^x]$

$\Rightarrow \dfrac{dy}{dx} = 2 \times \dfrac{1}{1+4^x} \times 2^x \log 2$ $\qquad \therefore \dfrac{dy}{dx} = \dfrac{2^{x+1}}{1+4^x}(\log 2)$.

Q18. Let $y = \cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$ $\Rightarrow y = \cot^{-1}\left(\dfrac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right)$

$\Rightarrow y = \cot^{-1}\left(\dfrac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right)$ $\qquad \left[\because 0 < x < \dfrac{\pi}{2} \Rightarrow 0 < \dfrac{x}{2} < \dfrac{\pi}{4}\right.$

$\Rightarrow y = \cot^{-1}\left(\dfrac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right)$ $\qquad \Rightarrow y = \cot^{-1}\left(\dfrac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \cot^{-1}\cot\dfrac{x}{2}$

$\Rightarrow y = \dfrac{x}{2}$ $\qquad \therefore \dfrac{dy}{dx} = \dfrac{1}{2}$.

Q19. Let $y = \tan^{-1}\left(\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right)$ $\qquad \Rightarrow y = \tan^{-1}\left(\sqrt{\dfrac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}\right)$ $\qquad \Rightarrow y = \tan^{-1}\left|\cot\dfrac{x}{2}\right|$

$\left[\because 0 < x < \pi \Rightarrow 0 < \dfrac{x}{2} < \dfrac{\pi}{2}\right]$ $\qquad \therefore y = \tan^{-1}\cot\dfrac{x}{2} = \tan^{-1}\tan\left(\dfrac{\pi}{2}-\dfrac{x}{2}\right)$

$\Rightarrow y = \dfrac{\pi}{2}-\dfrac{x}{2}$ $\qquad \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{2}$.

Q20. Let $y = \tan^{-1}\sqrt{\dfrac{1+\sin x}{1-\sin x}}$ $\qquad \Rightarrow y = \tan^{-1}\sqrt{\dfrac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}} = \tan^{-1}\left(\dfrac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)$

$\Rightarrow y = \tan^{-1}\left(\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$ $\qquad \Rightarrow y = \tan^{-1}\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right) = \dfrac{\pi}{4}+\dfrac{x}{2}$ $\qquad \therefore \dfrac{dy}{dx} = \dfrac{1}{2}$.

Q21. Let $y = \tan^{-1}\left(\dfrac{x-\sqrt{1-x^2}}{x+\sqrt{1-x^2}}\right)$ $\qquad$ Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x \ldots$(i)

$\Rightarrow y = \tan^{-1}\left(\dfrac{\cos\theta-\sqrt{1-\cos^2\theta}}{\cos\theta+\sqrt{1-\cos^2\theta}}\right)$ $\qquad \Rightarrow y = \tan^{-1}\left(\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$

$\Rightarrow y = \tan^{-1}\left(\dfrac{1-\tan\theta}{1+\tan\theta}\right) = \tan^{-1}\tan\left(\dfrac{\pi}{4}-\theta\right)$ $\qquad \Rightarrow y = \dfrac{\pi}{4}-\theta$

By using (i), we get : $y=\frac{\pi}{4}-\cos^{-1}x$ $\quad \therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$

Q22. Let $y=2\tan^{-1}\left[\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x\right]$

$\Rightarrow y=2\tan^{-1}\left[\operatorname{cosec}\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}-\tan\tan^{-1}\frac{1}{x}\right]$ $\Rightarrow y=2\tan^{-1}\left[\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right]$

$\Rightarrow y=2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ Put $x=\tan\theta\Rightarrow\theta=\tan^{-1}x\ \ldots$(i)

$\Rightarrow y=2\tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$ $\Rightarrow y=2\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$

$\Rightarrow y=2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]=2\tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$ $\Rightarrow y=2\tan^{-1}\left[\tan\frac{\theta}{2}\right]=\theta$

By (i), $y=\tan^{-1}x$ $\quad \therefore \frac{dy}{dx}=\frac{1}{1+x^2}$

Q23. Let $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ Putting $x=\tan\theta$, we get : $y=\tan^{-1}(\tan 2\theta)$

(a) If $-1<x<1$, then $x=\tan\theta \Rightarrow -1<\tan\theta<1 \Rightarrow -\frac{\pi}{4}<\theta<\frac{\pi}{4} \Rightarrow -\frac{\pi}{2}<2\theta<\frac{\pi}{2}$

$\therefore y=\tan^{-1}(\tan 2\theta)=2\theta$ $\quad\Rightarrow y=2\tan^{-1}x$ $\quad$ So, $\frac{dy}{dx}=\frac{2}{1+x^2}$

(b) If $-\infty<x<-1$, then $x=\tan\theta \Rightarrow -\infty<\tan\theta<-1 \Rightarrow -\frac{\pi}{2}<\theta<-\frac{\pi}{4} \Rightarrow -\pi<2\theta<-\frac{\pi}{2}$

$\therefore y=\tan^{-1}(\tan 2\theta)$ $\quad\Rightarrow y=\tan^{-1}(\tan(\pi+2\theta))=\pi+2\theta$ $\quad\left[\because -\pi<2\theta<-\frac{\pi}{2}\Rightarrow 0<\pi+2\theta<\frac{\pi}{2}\right.$

$\therefore y=\pi+2\tan^{-1}x$. So, $\frac{dy}{dx}=\frac{2}{1+x^2}$.

(c) If $1<x<\infty$, then $x=\tan\theta \Rightarrow 1<\tan\theta<\infty \Rightarrow \frac{\pi}{4}<\theta<\frac{\pi}{2} \Rightarrow \frac{\pi}{2}<2\theta<\pi$

$\therefore y=\tan^{-1}(\tan 2\theta)$ $\quad\Rightarrow y=\tan^{-1}(-\tan(\pi-2\theta))=-\tan^{-1}(\tan(\pi-2\theta))=-(\pi-2\theta)$

$\left[\because \frac{\pi}{2}<2\theta<\pi\Rightarrow -\frac{\pi}{2}<2\theta-\pi<0\right.$

$\therefore y=-\pi+2\tan^{-1}x$. So, $\frac{dy}{dx}=\frac{2}{1+x^2}$.

Q24. Let $y=\sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$ $\quad\Rightarrow y=\sin^{-1}\left(x\sqrt{1-[\sqrt{x}]^2}-\sqrt{x}\sqrt{1-x^2}\right)$

Put $x=\sin\theta$, $\sqrt{x}=\sin\beta\Rightarrow\theta=\sin^{-1}x$, $\beta=\sin^{-1}\sqrt{x}\ \ldots$(i)

$\Rightarrow y=\sin^{-1}\left(\sin\theta\sqrt{1-\sin^2\beta}-\sin\beta\sqrt{1-\sin^2\theta}\right)$ $\quad\Rightarrow y=\sin^{-1}(\sin\theta\cos\beta-\sin\beta\cos\theta)$

$\Rightarrow y=\sin^{-1}\sin(\theta-\beta)$ $\quad\Rightarrow y=\theta-\beta$ $\quad\Rightarrow y=\sin^{-1}x-\sin^{-1}\sqrt{x}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}}$ $\quad\therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x}\sqrt{1-x}}$

Q25. Given $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, $x^2\le 1$ Put $x^2=\cos 2\theta \Rightarrow \theta=\frac{1}{2}\cos^{-1}x^2$...(i)

$\Rightarrow y=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$ $\Rightarrow y=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$ $\Rightarrow y=\tan^{-1}\left(\frac{\tan(\pi/4)+\tan\theta}{1-\tan(\pi/4)\tan\theta}\right)$

$\Rightarrow y=\tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\pi}{4}+\theta$ $\Rightarrow y=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$ [By (i)

$\therefore \frac{dy}{dx}=0-\frac{1}{2}\times\frac{1}{\sqrt{1-x^4}}\times 2x=-\frac{x}{\sqrt{1-x^4}}$.

Q26. Let $y=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)=\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$ $\Rightarrow y=\pi-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow y=\pi-2\tan^{-1}x$ $\Rightarrow \frac{dy}{dx}=0-2\times\frac{1}{1+x^2}$ $\therefore \frac{dy}{dx}=-\frac{2}{1+x^2}$.

◘ Based On Derivatives Of Logarithmic Functions & Logarithmic Differentiation

Q01. Let $y=\log\left(x+\sqrt{x^2+a^2}\right)$ $\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\times\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\times\left(1+\frac{1}{2\sqrt{x^2+a^2}}[2x+0]\right)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\times\left(\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$ $\therefore \frac{dy}{dx}=\frac{1}{\sqrt{x^2+a^2}}$

Q02. Let $y=\log\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$ $\Rightarrow \frac{dy}{dx}=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}\times\frac{d}{dx}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$

$\Rightarrow \frac{dy}{dx}=\frac{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}\times\left\{\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)\times\frac{1}{2}\right\}$ $\Rightarrow \frac{dy}{dx}=\frac{1}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\sin 2\left(\frac{\pi}{4}+\frac{x}{2}\right)}$ $\Rightarrow \frac{dy}{dx}=\frac{1}{\sin\left(\frac{\pi}{2}+x\right)}=\frac{1}{\cos x}$ $\therefore \frac{dy}{dx}=\sec x$.

Q03. Let $y=\log\sqrt{\frac{1+\cot x}{1-\cot x}}$ $\Rightarrow y=\frac{1}{2}[\log(1+\cot x)-\log(1-\cot x)]$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[\frac{-\operatorname{cosec}^2x}{1+\cot x}-\frac{\operatorname{cosec}^2x}{1-\cot x}\right]$ $\Rightarrow \frac{dy}{dx}=-\frac{1}{2}\operatorname{cosec}^2x\left[\frac{1}{1+\cot x}+\frac{1}{1-\cot x}\right]$

$\Rightarrow \frac{dy}{dx}=-\frac{1}{2}\operatorname{cosec}^2x\left[\frac{1}{1+\cot x}+\frac{1}{1-\cot x}\right]=-\frac{\operatorname{cosec}^2x}{(1+\cot x)(1-\cot x)}$

$$\Rightarrow \frac{dy}{dx} = -\frac{\operatorname{cosec}^2 x}{1-\cot^2 x} = -\frac{\frac{1}{\sin^2 x}}{1-\frac{\cos^2 x}{\sin^2 x}} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x - \sin^2 x} = \sec 2x .$$

Q04. Let $y = \log\sqrt{\frac{a+b\sin x}{a-b\sin x}}$ $\quad \Rightarrow y = \frac{1}{2}[\log(a+b\sin x) - \log(a-b\sin x)]$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{b\cos x}{a+b\sin x} + \frac{b\cos x}{a-b\sin x}\right] \quad \Rightarrow \frac{dy}{dx} = \frac{b\cos x}{2}\left[\frac{1}{a+b\sin x} + \frac{1}{a-b\sin x}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{b\cos x}{2}\left[\frac{2a}{a^2-b^2\sin^2 x}\right] \quad \therefore \frac{dy}{dx} = \frac{ab\cos x}{(a^2-b^2\sin^2 x)} .$$

Q05. Let $y = \log\sqrt{\frac{1-\sin x}{1+\sin x}} + \log\left(\frac{e^x}{x^2}\right)$

$$\Rightarrow y = \frac{1}{2}[\log(1-\sin x) - \log(1+\sin x)] + \log e^x - \log x^2$$

$$\Rightarrow y = \frac{1}{2}[\log(1-\sin x) - \log(1+\sin x)] + x - 2\log x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{-\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x}\right] + 1 - \frac{2}{x} \quad \Rightarrow \frac{dy}{dx} = -\frac{\cos x}{2}\left[\frac{1}{1-\sin x} + \frac{1}{1+\sin x}\right] + 1 - \frac{2}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\cos x}{2}\left[\frac{2}{1-\sin^2 x}\right] + 1 - \frac{2}{x} \quad \therefore \frac{dy}{dx} = 1 - \sec x - \frac{2}{x}$$

Q06. Let $y = \log\frac{1+\tan x}{1-\tan x}$ $\quad \Rightarrow y = \log\tan\left(\frac{\pi}{4}+x\right)$ $\quad \Rightarrow \frac{dy}{dx} = \frac{1}{\tan\left(\frac{\pi}{4}+x\right)} \times \frac{d}{dx}\left[\tan\left(\frac{\pi}{4}+x\right)\right]$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos\left(\frac{\pi}{4}+x\right)}{\sin\left(\frac{\pi}{4}+x\right)} \times \sec^2\left(\frac{\pi}{4}+x\right) \times \frac{d}{dx}\left[\left(\frac{\pi}{4}+x\right)\right] \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\sin\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}+x\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{2\sin\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}+x\right)} \quad \Rightarrow \frac{dy}{dx} = \frac{2}{\sin 2\left(\frac{\pi}{4}+x\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin\left(\frac{\pi}{2}+2x\right)} = \frac{2}{\cos 2x} \quad \therefore \frac{dy}{dx} = 2\sec 2x$$

Q07. Let $y = \log\left(\frac{a+b\sin x}{a-b\sin x}\right)$ $\quad \Rightarrow y = \log(a+b\sin x) - \log(a-b\sin x)$

$$\Rightarrow \frac{dy}{dx} = \frac{b\cos x}{a+b\sin x} - \frac{-b\cos x}{a-b\sin x} \quad \Rightarrow \frac{dy}{dx} = b\cos x\left(\frac{1}{a+b\sin x} + \frac{1}{a-b\sin x}\right)$$

$$\Rightarrow \frac{dy}{dx} = b\cos x\left(\frac{2a}{a^2-b^2\sin^2 x}\right) \quad \therefore \frac{dy}{dx} = \frac{2ab\cos x}{a^2-b^2\sin^2 x}$$

Q08. Let $y = \log\sqrt{\frac{a+b\cos x}{a-b\cos x}}$ $\quad \Rightarrow y = \frac{1}{2}[\log(a+b\cos x) - \log(a-b\cos x)]$

$\Rightarrow y=\frac{1}{2}\left[\frac{-b\sin x}{a+b\cos x}-\frac{-b(-\sin x)}{a-b\cos x}\right]$ $\Rightarrow y=\frac{-b\sin x}{2}\left[\frac{1}{a+b\cos x}+\frac{1}{a-b\cos x}\right]$

$\Rightarrow y=\frac{-b\sin x}{2}\left[\frac{2a}{a^2-b^2\cos^2 x}\right]$ $\therefore y=\frac{ab\sin x}{b^2\cos^2 x-a^2}$

Q09. Let $y=\frac{\sin x-x\cos x}{x\sin x+\cos x}+\log\left(x+\sqrt{x^2-a^2}\right)$

$$\Rightarrow \frac{dy}{dx}=\frac{(x\sin x+\cos x)(\sin x-x\cos x)'-(\sin x-x\cos x)(x\sin x+\cos x)'}{(x\sin x+\cos x)^2}+\frac{1}{x+\sqrt{x^2-a^2}}\times\frac{d}{dx}\left(x+\sqrt{x^2-a^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{(x\sin x+\cos x)(\cos x+x\sin x-\cos x)-(\sin x-x\cos x)(x\cos x+\sin x-\sin x)}{(x\sin x+\cos x)^2}+\frac{1}{x+\sqrt{x^2-a^2}}\times\left(1+\frac{2x}{2\sqrt{x^2-a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{(x\sin x+\cos x)(x\sin x)-(\sin x-x\cos x)(x\cos x)}{(x\sin x+\cos x)^2}+\frac{1}{x+\sqrt{x^2-a^2}}\times\left(\frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{x\left[(x\sin^2 x+\cos x\sin x-\sin x\cos x+x\cos^2 x\right]}{(x\sin x+\cos x)^2}+\frac{1}{\sqrt{x^2-a^2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x\left[x(\sin^2 x+\cos^2 x)\right]}{(x\sin x+\cos x)^2}+\frac{1}{\sqrt{x^2-a^2}} \quad \therefore \frac{dy}{dx}=\frac{x^2}{(x\sin x+\cos x)^2}+\frac{1}{\sqrt{x^2-a^2}}$$

Q10. Let $y=\log\tan\sqrt{x+1}+\frac{5x}{\sqrt[3]{1-x^2}}+\sin^2(2x-3)$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\tan\sqrt{x+1}}\times\frac{d}{dx}\left[\tan\sqrt{x+1}\right]+\frac{du}{dx}+2\sin(2x-3)\times\frac{d}{dx}\left[\sin(2x-3)\right],\text{ where } u=\frac{5x}{\sqrt[3]{1-x^2}}$$

Now, $u=\frac{5x}{\sqrt[3]{1-x^2}}$ $\Rightarrow \log u=\log\left(\frac{5x}{\sqrt[3]{1-x^2}}\right)$ $\Rightarrow \log u=\log 5x-\frac{1}{3}\log(1-x^2)$

$\therefore \frac{1}{u}\times\frac{du}{dx}=\frac{5}{5x}-\frac{1}{3}\left(\frac{-2x}{1-x^2}\right)$ $\Rightarrow \frac{du}{dx}=u\left(\frac{1}{x}+\frac{2x}{3-3x^2}\right)$ $\Rightarrow \frac{du}{dx}=\frac{5x}{\sqrt[3]{1-x^2}}\left(\frac{3-x^2}{3x(1-x^2)}\right)$

$$\text{So, } \frac{dy}{dx}=\frac{1}{\tan\sqrt{x+1}}\times\left[\sec^2\sqrt{x+1}\right]\times\frac{1}{2\sqrt{x+1}}+\frac{5x}{\sqrt[3]{1-x^2}}\left(\frac{3-x^2}{3x(1-x^2)}\right)+2\sin(2x-3)\times\cos(2x-3)\times 2$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sin\sqrt{x+1}\cos\sqrt{x+1}}\times\frac{1}{\sqrt{x+1}}+\frac{5x}{\sqrt[3]{1-x^2}}\left(\frac{3-x^2}{3x(1-x^2)}\right)+2\sin 2(2x-3)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sin 2\sqrt{x+1}}\times\frac{1}{\sqrt{x+1}}+\frac{5x}{\sqrt[3]{1-x^2}}\left(\frac{3-x^2}{3x(1-x^2)}\right)+2\sin(4x-6)$$

$$\therefore \frac{dy}{dx}=\frac{\operatorname{cosec}2\sqrt{x+1}}{\sqrt{x+1}}+\frac{5x}{\sqrt[3]{1-x^2}}\left[\frac{3-x^2}{3x(1-x^2)}\right]+2\sin(4x-6)$$

Q11. Let $y=\log\left(\frac{x^2+x+1}{x^2-x+1}\right)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{1-x^2}\right)$

$$\Rightarrow y=\log(x^2+x+1)-\log(x^2-x+1)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x+1}{x^2+x+1}-\frac{2x-1}{x^2-x+1}+\frac{2}{\sqrt{3}}\times\frac{1}{1+\left(\frac{\sqrt{3}x}{1-x^2}\right)^2}\times\frac{d}{dx}\left(\frac{\sqrt{3}x}{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{(2x+1)(x^2-x+1)-(2x-1)(x^2+x+1)}{(x^2+x+1)(x^2-x+1)}+\frac{2}{\sqrt{3}}\times\frac{(1-x^2)^2}{1-2x^2+x^4+3x^2}\times\left(\frac{(1-x^2)\sqrt{3}-\sqrt{3}x(-2x)}{(1-x^2)^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x^3-2x^2+2x+x^2-x+1-(2x^3+2x^2+2x-x^2-x-1)}{(x^2+1+x)(x^2+1-x)}+\frac{2}{\sqrt{3}}\times\frac{1}{1+x^2+x^4}\times\left((1-x^2)\sqrt{3}+2\sqrt{3}x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2-2x^2}{x^4+1+x^2}+\frac{2}{\sqrt{3}}\times\frac{1}{x^4+1+x^2}\times\left((1+x^2)\sqrt{3}\right)$$

$$\Rightarrow \frac{dy}{dx}=2\left[\frac{1-x^2}{x^4+1+x^2}+\frac{1+x^2}{x^4+1+x^2}\right] \qquad \therefore \frac{dy}{dx}=\frac{4}{x^4+x^2+1}$$

Q12. Let $y=\frac{1}{2\sqrt{2}}\log\left(\frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2}\right)+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{1-x^2}\right)$

$$\Rightarrow y=\frac{1}{2\sqrt{2}}\left[\log(1+\sqrt{2}x+x^2)-\log(1-\sqrt{2}x+x^2)\right]+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{2}}\left[\frac{2x+\sqrt{2}}{1+\sqrt{2}x+x^2}-\frac{2x-\sqrt{2}}{1-\sqrt{2}x+x^2}\right]+\frac{1}{\sqrt{2}}\times\frac{1}{1+\left(\frac{\sqrt{2}x}{1-x^2}\right)^2}\times\frac{d}{dx}\left(\frac{\sqrt{2}x}{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{2}}\left[\frac{(2x+\sqrt{2})(1-\sqrt{2}x+x^2)-(2x-\sqrt{2})(1+\sqrt{2}x+x^2)}{(1+x^2+\sqrt{2}x)(1+x^2-\sqrt{2}x)}\right]+\frac{1}{\sqrt{2}}\times\frac{(1-x^2)^2}{1-2x^2+x^4+2x^2}\times\left(\frac{(1-x^2)\sqrt{2}-\sqrt{2}x(-2x)}{(1-x^2)^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[\frac{(\sqrt{2}x+1)(1-\sqrt{2}x+x^2)-(\sqrt{2}x-1)(1+\sqrt{2}x+x^2)}{(1+x^2+\sqrt{2}x)(1+x^2-\sqrt{2}x)}\right]+\frac{1}{1+x^4}\times\left((1-x^2)+2x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[\frac{\sqrt{2}x-2x^2+\sqrt{2}x^3+1-\sqrt{2}x+x^2-(\sqrt{2}x+2x^2+\sqrt{2}x^3-1-\sqrt{2}x-x^2)}{(1+x^2)^2-(\sqrt{2}x)^2}\right]+\frac{1+x^2}{1+x^4}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[\frac{2-2x^2}{1+2x^2+x^4-2x^2}\right]+\frac{1+x^2}{1+x^4} \qquad \Rightarrow \frac{dy}{dx}=\frac{1-x^2}{1+x^4}+\frac{1+x^2}{1+x^4}=\frac{2}{1+x^4}$$

Q13. Let $y=\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ Taking logarithm on both the sides, we get :

$$\Rightarrow \log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \qquad \Rightarrow \log y=\frac{1}{2}\log\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}$$

$$\Rightarrow \log y=\frac{1}{2}[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)]$$

Differentiating w.r.t. x both the sides, we get : $\dfrac{1}{y}\times\dfrac{dy}{dx}=\dfrac{1}{2}\left[\dfrac{1}{x-1}+\dfrac{1}{x-2}-\dfrac{1}{x-3}-\dfrac{1}{x-4}-\dfrac{1}{x-5}\right]$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left\{\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right\}$$

Q14. Let $y=u+v$, where $u=\left(x+\dfrac{1}{x}\right)^x$ and $v=x^{x+\frac{1}{x}}$

On differentiating w.r.t. x both the sides, $\dfrac{dy}{dx}=\dfrac{du}{dx}+\dfrac{dv}{dx}$ …(i)

Now $u=\left(x+\dfrac{1}{x}\right)^x \qquad \Rightarrow \log u=\log\left(x+\dfrac{1}{x}\right)^x \qquad \Rightarrow \log u=x\log\left(x+\dfrac{1}{x}\right)$

$$\Rightarrow \frac{1}{u}\times\frac{du}{dx}=x\times\frac{1}{x+\frac{1}{x}}\times\frac{d}{dx}\left(x+\frac{1}{x}\right)+\log\left(x+\frac{1}{x}\right)\times\frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{u}\times\frac{du}{dx}=x\times\frac{x}{x^2+1}\times\left(1-\frac{1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)\times 1$$

$$\Rightarrow \frac{1}{u}\times\frac{du}{dx}=\frac{x^2}{x^2+1}\times\left(\frac{x^2-1}{x^2}\right)+\log\left(x+\frac{1}{x}\right) \qquad \Rightarrow \frac{du}{dx}=u\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]$$

And, $v=x^{x+\frac{1}{x}} \qquad \Rightarrow \log v=\log\left(x^{x+\frac{1}{x}}\right) \qquad \Rightarrow \log v=\left(x+\dfrac{1}{x}\right)\log x$

$$\Rightarrow \frac{1}{v}\times\frac{dv}{dx}=\left(x+\frac{1}{x}\right)\frac{1}{x}+\log x\times\left(1-\frac{1}{x^2}\right) \qquad \Rightarrow \frac{dv}{dx}=v\left[\frac{x^2+1}{x^2}+\log x\times\left(\frac{x^2-1}{x^2}\right)\right]$$

$$\Rightarrow \frac{dv}{dx}=x^{x+\frac{1}{x}}\left[\frac{x^2+1+(x^2-1)\log x}{x^2}\right]$$

Substituting du/dx and dv/dx in (i), we get :

$$\therefore \frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{x+\frac{1}{x}}\left[\frac{x^2+1+(x^2-1)\log x}{x^2}\right]$$

Q15. Let $y=x^{\cos^{-1}x}+\cos^{-1}\sqrt{x} \qquad \Rightarrow y=e^{\log x^{\cos^{-1}x}}+\cos^{-1}\sqrt{x} \qquad \Rightarrow y=e^{\cos^{-1}x\,\log x}+\cos^{-1}\sqrt{x}$

$$\Rightarrow \frac{dy}{dx}=e^{\cos^{-1}x\,\log x}\times\frac{d}{dx}\left[\cos^{-1}x\ \log x\right]-\frac{1}{\sqrt{1-(\sqrt{x})^2}}\times\frac{d}{dx}\left[\sqrt{x}\right]$$

$$\Rightarrow \frac{dy}{dx}=x^{\cos^{-1}x}\times\left[\frac{\cos^{-1}x}{x}+\log x\left(-\frac{1}{\sqrt{1-x^2}}\right)\right]-\frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = x^{\cos^{-1}x}\left[\frac{\cos^{-1}x}{x} - \frac{\log x}{\sqrt{1-x^2}}\right] - \frac{1}{2\sqrt{x-x^2}}$$

Q16. Let $y = x^{\sin x - \cos x} + \log\left(x^x + \sec^2 x\right)$ $\Rightarrow y = e^{\log x^{\sin x - \cos x}} + \log\left(x^x + \sec^2 x\right)$

$\Rightarrow y = e^{(\sin x - \cos x)\log x} + \log\left(x^x + \sec^2 x\right)$ On diff. w.r.t. x both the sides, we get :

$$\Rightarrow \frac{dy}{dx} = e^{(\sin x - \cos x)\log x}\left(\frac{(\sin x - \cos x)}{x} + \log x(\cos x + \sin x)\right) + \frac{1}{x^x + \sec^2 x} \times \frac{d}{dx}\left(x^x + \sec^2 x\right)$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x - \cos x}\left(\frac{\sin x - \cos x}{x} + (\cos x + \sin x)\log x\right) + \frac{1}{x^x + \sec^2 x} \times \left[\frac{d}{dx}(x^x) + \frac{d}{dx}(\sec^2 x)\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x - \cos x}\left(\frac{\sin x - \cos x}{x} + (\cos x + \sin x)\log x\right)$$

$$+ \frac{1}{x^x + \sec^2 x} \times \left[x^x(1 + \log x) + 2\sec x(\sec x \tan x)\right]$$

$$\therefore \frac{dy}{dx} = x^{\sin x - \cos x}\left(\frac{\sin x - \cos x}{x} + (\cos x + \sin x)\log x\right) + \frac{x^x(1 + \log x) + 2\tan x \sec^2 x}{x^x + \sec^2 x}$$

Q17. Let $y = \sin(x^x) + (\cos x)^x$ $\Rightarrow y = \sin(x^x) + e^{\log(\cos x)^x}$ $\Rightarrow y = \sin(x^x) + e^{x\log(\cos x)}$

$$\Rightarrow \frac{dy}{dx} = \cos(x^x) \times \frac{d}{dx}(x^x) + e^{x\log(\cos x)}\left[x \times \frac{-\sin x}{\cos x} + \log\cos x.(1)\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos(x^x) \times x^x(1 + \log x) + \cos x^x\left[\log\cos x - x\tan x\right]$$

$$\therefore \frac{dy}{dx} = x^x(1 + \log x)\cos(x^x) + \cos x^x\left[\log\cos x - x\tan x\right]$$

Q18. Let $y = x^{\log x} + \left(\log x\right)^x$ $\Rightarrow y = e^{\log x^{\log x}} + e^{\log(\log x)^x}$

$\Rightarrow y = e^{\log x \log x} + e^{x\log(\log x)}$ $\Rightarrow y = e^{(\log x)^2} + e^{x\log(\log x)}$

On differentiating w.r.t. x both sides, we get :

$$\Rightarrow \frac{dy}{dx} = e^{(\log x)^2}\left[2\log x \times \frac{1}{x}\right] + e^{x\log(\log x)}\left[x \times \frac{1}{\log x} \times \frac{1}{x} + \log(\log x).(1)\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{\log x}\left[\frac{2\log x}{x}\right] + (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right]$$

$$\therefore \frac{dy}{dx} = 2x^{\log x - 1}\log x + \left(\log x\right)^x\left[\frac{1}{\log x} + \log\log x\right]$$

Q19. Let $y = e^{\cos^{-1}x} + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ $\Rightarrow y = e^{\cos^{-1}x} + 2\tan^{-1}x$

$\Rightarrow \frac{dy}{dx} = e^{\cos^{-1}x} \times \frac{d}{dx}(\cos^{-1}x) + 2 \times \frac{1}{1+x^2}$ $\therefore \frac{dy}{dx} = \frac{2}{1+x^2} - \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}$

Q20. Let $y = x^{\tan x} + \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\Rightarrow y = e^{\log x^{\tan x}} + \frac{e^{2x} - 1}{e^{2x} + 1}$ $\Rightarrow y = e^{\tan x \log x} + 1 - \frac{2}{e^{2x} + 1}$

$$\Rightarrow \frac{dy}{dx} = e^{\tan x \log x}\left(\tan x \times \frac{1}{x} + \log x \sec^2 x\right) + 0 - \left(-\frac{2}{(e^{2x}+1)^2} \times e^{2x} \times 2\right)$$

$$\Rightarrow \frac{dy}{dx} = x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x\ \log x\right) + \frac{4e^{2x}}{(e^{2x}+1)^2}$$

or, $\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \log x \sec^2 x\right] + \frac{4}{(e^x + e^{-x})^2}$

Q21. Let $y = (x\cos x)^x + (x\sin x)^{1/x}$ $\Rightarrow y = e^{\log(x\cos x)^x} + e^{\log(x\sin x)^{1/x}}$

$\Rightarrow y = e^{x\log(x\cos x)} + e^{\frac{1}{x}\log(x\sin x)}$

$\Rightarrow \frac{dy}{dx} = e^{x\log(x\cos x)}\left[x \times \frac{1}{x\cos x}(-x\sin x + \cos x) + \log(x\cos x).1\right]$

$+ e^{\frac{1}{x}\log(x\sin x)}\left[\frac{1}{x} \times \frac{1}{x\sin x}(x\cos x + \sin x) + \log(x\sin x)\left(-\frac{1}{x^2}\right)\right]$

$\Rightarrow \frac{dy}{dx} = (x\cos x)^x\left[\frac{\cos x - x\sin x}{\cos x} + \log(x\cos x)\right] + (x\sin x)^{1/x}\left[\frac{x\cos x + \sin x}{x^2\sin x} - \frac{\log(x\sin x)}{x^2}\right]$

$\therefore \frac{dy}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{1/x}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$

Q22. Let $y = \log\left(x^x + \sin x^{\cos x}\right)$ $\Rightarrow y = \log(u+v)$, where $u = x^x$ and $v = \sin x^{\cos x}$

$\therefore \frac{dy}{dx} = \frac{1}{u+v}\left(\frac{du}{dx} + \frac{dv}{dx}\right) \ldots(i)$

Now $u = x^x$ and $v = \sin x^{\cos x}$ $\Rightarrow u = e^{\log x^x}$ and $v = e^{\log \sin x^{\cos x}}$ $\Rightarrow u = e^{x\log x}$ and $v = e^{\cos x \log \sin x}$

$\Rightarrow \frac{du}{dx} = e^{x\log x}\left(x \times \frac{1}{x} + \log x.1\right)$ and $\frac{dv}{dx} = e^{\cos x\log\sin x}\left(\cos x \times \frac{\cos x}{\sin x} + \log\sin x(-\sin x)\right)$

$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$ and $\frac{dv}{dx} = \sin x^{\cos x}(\cos x\cot x - \sin x\log\sin x)$

Substituting the value of du/dx and dv/dx in (i), we get :

$\frac{dy}{dx} = \frac{1}{x^x + \sin x^{\cos x}}\left(x^x(1+\log x) + \sin x^{\cos x}(\cos x\cot x - \sin x\log\sin x)\right)$

i.e., $\frac{dy}{dx} = \frac{x^x(1+\log x) + \sin x^{\cos x}(\cos x\cot x - \sin x\log\sin x)}{x^x + \sin x^{\cos x}}$

Q23. Let $y = e^{-ax^2}\sin(x\log x)$ $\Rightarrow \frac{dy}{dx} = e^{-ax^2}\frac{d}{dx}[\sin(x\log x)] + \sin(x\log x)\frac{d}{dx}\left[e^{-ax^2}\right]$

$\Rightarrow \frac{dy}{dx} = e^{-ax^2}\cos(x\log x)\left[x \times \frac{1}{x} + \log x.1\right] + \sin(x\log x) \times e^{-ax^2} \times (-2ax)$

$\therefore \frac{dy}{dx} = e^{-ax^2}\left[\cos(x\log x)(1+\log x) - 2a\,x\sin(x\log x)\right]$

Q24. Let $y = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log(x + \sqrt{a^2+x^2})$ $\Rightarrow 2y = x\sqrt{a^2+x^2} + a^2\log(x + \sqrt{a^2+x^2})$

$\Rightarrow 2\frac{dy}{dx} = x \times \frac{1}{2\sqrt{a^2+x^2}}(0+2x) + \sqrt{a^2+x^2} \times 1 + a^2 \times \frac{1}{x+\sqrt{a^2+x^2}}\left(1 + \frac{1}{2\sqrt{a^2+x^2}}(0+2x)\right)$

$\Rightarrow 2\frac{dy}{dx} = \frac{x^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2} + a^2 \times \frac{1}{x+\sqrt{a^2+x^2}}\left(\frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}\right)$

$\Rightarrow 2\frac{dy}{dx} = \frac{x^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2} + \frac{a^2}{\sqrt{a^2+x^2}}$

$\Rightarrow 2\frac{dy}{dx} = \frac{x^2+a^2}{\sqrt{a^2+x^2}} + \sqrt{a^2+x^2} = \sqrt{a^2+x^2} + \sqrt{a^2+x^2}$ $\therefore \frac{dy}{dx} = \sqrt{a^2+x^2}$

Q25. Let $y=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log(x+\sqrt{x^2-a^2})$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{2}\times\frac{1}{2\sqrt{x^2-a^2}}(2x-0)+\sqrt{x^2-a^2}\times\frac{1}{2}-\frac{a^2}{2}\times\frac{1}{x+\sqrt{x^2-a^2}}\left(1+\frac{1}{2\sqrt{x^2-a^2}}(2x-0)\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2}{2\sqrt{x^2-a^2}}+\frac{\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\times\frac{1}{x+\sqrt{x^2-a^2}}\left(\frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^2}{2\sqrt{x^2-a^2}}+\frac{\sqrt{x^2-a^2}}{2}-\frac{a^2}{2\sqrt{x^2-a^2}}=\frac{x^2-a^2}{2\sqrt{x^2-a^2}}+\frac{\sqrt{x^2-a^2}}{2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x^2-a^2}}{2}+\frac{\sqrt{x^2-a^2}}{2} \qquad \therefore \frac{dy}{dx}=\sqrt{x^2-a^2}$$

Q26. Let $y=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{2}\times\frac{1}{2\sqrt{a^2-x^2}}(-2x)+\sqrt{a^2-x^2}\times\frac{1}{2}+\frac{a^2}{2}\times\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\times\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\times\frac{a}{\sqrt{a^2-x^2}}\times\left(\frac{1}{a}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}=\frac{a^2-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{a^2-x^2}}{2}+\frac{\sqrt{a^2-x^2}}{2}=\sqrt{a^2-x^2}$$

Q27. Let $y=u+v$, where $u=\frac{5x}{\sqrt[3]{(1-x)^2}}$ and $v=\tan^{-1}(\log_{10}x)$ $\qquad \therefore \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\ldots$(i)

Now $u=\frac{5x}{\sqrt[3]{(1-x)^2}}$ and $v=\tan^{-1}(\log_{10}x)$

$$\Rightarrow \log u=\log\left(\frac{5x}{\sqrt[3]{(1-x)^2}}\right) \text{ and } \frac{dv}{dx}=\frac{1}{1+(\log_{10}x)^2}\times\frac{d}{dx}(\log_{10}x)$$

$$\Rightarrow \log u=\log 5x-\frac{2}{3}\log(1-x) \text{ and } \frac{dv}{dx}=\frac{1}{1+(\log_{10}x)^2}\times\frac{d}{dx}\left[\frac{\log_e x}{\log_e 10}\right]$$

$$\Rightarrow \frac{1}{u}\times\frac{du}{dx}=\frac{5}{5x}-\frac{2}{3}\times\frac{1}{1-x}(0-1) \text{ and } \frac{dv}{dx}=\frac{1}{1+(\log_{10}x)^2}\times\frac{1}{x\log_e 10}$$

$$\Rightarrow \frac{du}{dx}=u\left[\frac{1}{x}+\frac{2}{3}\times\frac{1}{1-x}\right] \text{ and } \frac{dv}{dx}=\frac{1}{x\log_e 10\left[1+(\log_{10}x)^2\right]}$$

$$\Rightarrow \frac{du}{dx}=\frac{5x}{\sqrt[3]{(1-x)^2}}\left[\frac{3-3x+2x}{3x(1-x)}\right] \text{ and } \frac{dv}{dx}=\frac{\log_{10}e}{x\left[1+(\log_{10}x)^2\right]}$$

$$\Rightarrow \frac{du}{dx}=\frac{5(3-x)}{3\sqrt[3]{(1-x)^5}} \text{ and } \frac{dv}{dx}=\frac{\log_{10}e}{x\left[1+(\log_{10}x)^2\right]}$$

Substituting values of du/dx and dv/dx in (i), we get : $\frac{dy}{dx}=\frac{5(3-x)}{3(1-x)^{5/3}}+\frac{\log_{10} e}{x\left[1+(\log_{10} x)^2\right]}$

Q28. Let $y=x^{\cos^{-1}(x-1)}+\frac{x-1}{x+1}$ $\Rightarrow y=e^{\log x^{\cos^{-1}(x-1)}}+\frac{x+1-2}{x+1}$ $\Rightarrow y=e^{\cos^{-1}(x-1)\log x}+1-\frac{2}{x+1}$

$$\Rightarrow \frac{dy}{dx}=e^{\cos^{-1}(x-1)\log x}\left[\cos^{-1}(x-1)\times\frac{1}{x}+\log x\left(-\frac{1}{\sqrt{1-(x-1)^2}}\times(1-0)\right)\right]+0-\left(\frac{-2}{(x+1)^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=e^{\cos^{-1}(x-1)\log x}\left[\frac{\cos^{-1}(x-1)}{x}-\frac{\log x}{\sqrt{1-(x-1)^2}}\right]+\frac{2}{(x+1)^2}$$

$$\therefore \frac{dy}{dx}=x^{\cos^{-1}(x-1)}\left[\frac{\cos^{-1}(x-1)}{x}-\frac{\log x}{\sqrt{2x-x^2}}\right]+\frac{2}{(x+1)^2}$$

Q29. (a) We have $x^y+y^x=a^b$ $\Rightarrow e^{\log x^y}+e^{\log y^x}=a^b$ $\Rightarrow e^{y\log x}+e^{x\log y}=a^b$

$$\Rightarrow e^{y\log x}\left(y\times\frac{1}{x}+\log x\times\frac{dy}{dx}\right)+e^{x\log y}\left(x\times\frac{1}{y}\times\frac{dy}{dx}+\log y.1\right)=0$$

$$\Rightarrow x^y\left(\frac{y}{x}+\log x\times\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}\times\frac{dy}{dx}+\log y\right)=0$$

$$\Rightarrow x^{y-1}y+x^y\log x\times\frac{dy}{dx}+y^{x-1}x\times\frac{dy}{dx}+y^x\log y=0$$

$$\Rightarrow \left(x^y\log x+y^{x-1}x\right)\frac{dy}{dx}=-\left(y^x\log y+x^{y-1}y\right) \qquad \therefore \frac{dy}{dx}=-\left(\frac{y^x\log y+x^{y-1}y}{x^y\log x+y^{x-1}x}\right)$$

(b) We have $x^y+y^x+x^x=m^n \Rightarrow e^{\log x^y}+e^{\log y^x}+e^{\log x^x}=m^n \Rightarrow e^{y\log x}+e^{x\log y}+e^{x\log x}=m^n$

$$\Rightarrow e^{y\log x}\left(y\times\frac{1}{x}+\log x\times\frac{dy}{dx}\right)+e^{x\log y}\left(x\times\frac{1}{y}\times\frac{dy}{dx}+\log y.1\right)+e^{x\log x}\left(x\times\frac{1}{x}+\log x.1\right)=\frac{d}{dx}\left(m^n\right)$$

$$\Rightarrow x^y\left(\frac{y}{x}+\log x\times\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}\times\frac{dy}{dx}+\log y\right)+x^x\left(1+\log x\right)=0$$

$$\Rightarrow \left(x^{y-1}y+x^y\log x\times\frac{dy}{dx}\right)+\left(y^{x-1}x\times\frac{dy}{dx}+y^x\log y\right)+x^x\left(1+\log x\right)=0$$

$$\Rightarrow \left(x^y\log x+y^{x-1}x\right)\times\frac{dy}{dx}=-\left[x^{y-1}y+y^x\log y+x^x\left(1+\log x\right)\right]$$

$$\therefore \frac{dy}{dx}=-\left[\frac{x^{y-1}y+y^x\log y+x^x\left(1+\log x\right)}{x^y\log x+y^{x-1}x}\right]$$

Q30. Let $(\tan^{-1}x)^y+y^{\cot x}=1$ $\Rightarrow e^{y\log(\tan^{-1}x)}+e^{\cot x\log y}=1$

$$\Rightarrow e^{y\log(\tan^{-1}x)}\left(y\times\frac{1}{\tan^{-1}x}\times\frac{1}{1+x^2}+\log(\tan^{-1}x)\times\frac{dy}{dx}\right)$$

$$+e^{\cot x\log y}\left(\cot x\times\frac{1}{y}\times\frac{dy}{dx}-\operatorname{cosec}^2 x\log y\right)=0$$

$$\Rightarrow (\tan^{-1}x)^y\left(\frac{y}{(1+x^2)\tan^{-1}x}+\log(\tan^{-1}x)\times\frac{dy}{dx}\right)+y^{\cot x}\left(\frac{\cot x}{y}\times\frac{dy}{dx}-\operatorname{cosec}^2 x\log y\right)=0$$

$$\Rightarrow \frac{y(\tan^{-1}x)^{y-1}}{(1+x^2)}+(\tan^{-1}x)^y\log(\tan^{-1}x)\times\frac{dy}{dx}+y^{\cot x-1}\cot x\times\frac{dy}{dx}-y^{\cot x}\operatorname{cosec}^2 x\log y=0$$

$\therefore \dfrac{dy}{dx} = \dfrac{y^{\cot x}\operatorname{cosec}^2 x \log y - y(\tan^{-1}x)^{y-1}(1+x^2)^{-1}}{(\tan^{-1}x)^y \log(\tan^{-1}x) + y^{\cot x - 1}\cot x}$.

Q31. We have $y = (1+\log x)^x + x^{x\cos x}$.

Let $y = u + v$ where $u = (\log x)^x,\ v = x^{x\cos x}$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(i)

Now $u = (1+\log x)^x$ $\Rightarrow \log u = \log(1+\log x)^x$ $\Rightarrow \log u = x\log(1+\log x)$

$\Rightarrow \dfrac{1}{u}\times\dfrac{du}{dx} = x\times\dfrac{1}{1+\log x}\times\dfrac{1}{x} + \log(1+\log x)\times 1$

$\Rightarrow \dfrac{du}{dx} = (1+\log x)^x\left(\dfrac{1}{1+\log x} + \log(1+\log x)\right)$

And, $v = x^{x\cos x}$ $\Rightarrow \log v = \log(x)^{x\cos x}$ $\Rightarrow \log v = x\cos x\log x$

$\Rightarrow \dfrac{1}{v}\times\dfrac{dv}{dx} = x\cos x\times\dfrac{1}{x} + \log x\left(\cos x - x\sin x\right)$

$\Rightarrow \dfrac{dv}{dx} = x^{x\cos x}\left[\cos x + \log x\left(\cos x - x\sin x\right)\right]$

By (i), $\dfrac{dy}{dx} = (1+\log x)^x\left(\dfrac{1}{1+\log x} + \log(1+\log x)\right) + x^{x\cos x}\left[\cos x(1+\log x) - x\sin x\log x\right]$

Q32. Given $y = x^{e^{-x^2}}$ $\Rightarrow \log y = e^{-x^2}\log x$ $\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{e^{-x^2}}{x} - 2xe^{-x^2}\log x$

$\Rightarrow \dfrac{dy}{dx} = y\left(\dfrac{e^{-x^2}}{x} - 2xe^{-x^2}\log x\right)$ $\therefore \dfrac{dy}{dx} = x^{e^{-x^2}}e^{-x^2}\left(\dfrac{1}{x} - 2x\log x\right)$

Q33. Let $y = \cos^{-1}\left[\sin\sqrt{\dfrac{1+x}{2}}\right] + x^x$ $\Rightarrow y = \cos^{-1}\left[\cos\left(\dfrac{\pi}{2} - \dfrac{\sqrt{1+x}}{\sqrt{2}}\right)\right] + e^{\log x^x}$

$\Rightarrow y = \dfrac{\pi}{2} - \dfrac{\sqrt{1+x}}{\sqrt{2}} + e^{x\log x}$ $\Rightarrow \dfrac{dy}{dx} = 0 - \dfrac{1}{\sqrt{2}}\times\dfrac{1}{2\sqrt{1+x}} + e^{x\log x}\left(x\times\dfrac{1}{x} + \log x\right)$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{2\sqrt{2}\sqrt{1+x}} + x^x\left(1+\log x\right)$ $\therefore \left(\dfrac{dy}{dx}\right)_{\text{at } x=1} = -\dfrac{1}{2\sqrt{2}\sqrt{1+1}} + 1^1\left(1+\log 1\right) = \dfrac{3}{4}$.

◘ Based On Derivatives Of Implicit Functions

Q01. Given $x = y\log\left(xy\right)$ ...(i) $\Rightarrow x = y\left[\log x + \log y\right]$

$\Rightarrow \dfrac{dx}{dx} = y\left[\dfrac{1}{x} + \dfrac{1}{y}\times\dfrac{dy}{dx}\right] + (\log x + \log y)\dfrac{dy}{dx}$ $\Rightarrow 1 = \dfrac{y}{x} + \dfrac{dy}{dx} + \log(xy)\dfrac{dy}{dx}$

$\Rightarrow 1 - \dfrac{y}{x} = \dfrac{dy}{dx}\left[1 + \log(xy)\right]$ $\Rightarrow \dfrac{x-y}{x} = \dfrac{dy}{dx}\left[1 + \dfrac{x}{y}\right]$ [By (i)]

$\Rightarrow \dfrac{x-y}{x} = \dfrac{dy}{dx}\left[\dfrac{x+y}{y}\right]$ $\therefore \dfrac{dy}{dx} = \dfrac{(x-y)y}{(x+y)x}$

Q02. We've $x^3 + x^2y + \sin xy = 81$ $\Rightarrow 3x^2 + x^2y' + y(2x) + \cos xy\left[xy' + y.1\right] = 0$

$\Rightarrow 3x^2 + x^2y' + y(2x) + \cos xy\left[xy' + y.1\right] = 0$ $\Rightarrow 3x^2 + x^2y' + 2xy + xy'\cos xy + y\cos xy = 0$

$\Rightarrow y'(x^2 + x\cos xy) = -(3x^2 + 2xy + y\cos xy)$ $\therefore \dfrac{dy}{dx} = -\left(\dfrac{3x^2 + 2xy + y\cos xy}{x^2 + x\cos xy}\right)$

Q03. We've $x\cos y + y\sin x = \tan(x+y)$

$\Rightarrow x(-\sin y)y' + \cos y.1 + y\cos x + y'\sin x = \sec^2(x+y)(1+y')$

$\Rightarrow -xy'\sin y + \cos y + y\cos x + y'\sin x = \sec^2(x+y) + y'\sec^2(x+y)$

$\Rightarrow \cos y + y\cos x - \sec^2(x+y) = y'\left[\sec^2(x+y) - \sin x + x\sin y\right]$

$\Rightarrow \frac{dy}{dx} = \frac{\cos y + y\cos x - \sec^2(x+y)}{x\sin y + \sec^2(x+y) - \sin x}$

Q04. We have $x^m y^n = (x-y)^{m+n}$ $\Rightarrow \log(x^m y^n) = \log(x-y)^{m+n}$

$\Rightarrow m\log x + n\log y = (m+n)\log(x-y)$ Differentiating w.r.t. x both sides, we get :

$\Rightarrow m\times\frac{1}{x} + n\times\frac{1}{y}\times y' = (m+n)\times\frac{1}{x-y}\times(1-y')$

$\Rightarrow \frac{m}{x} + \frac{n}{y}\times y' = \frac{m+n}{x-y} - y'\frac{m+n}{x-y}$ $\Rightarrow \frac{n}{y}\times y' + y'\frac{m+n}{x-y} = \frac{m+n}{x-y} - \frac{m}{x}$

$\Rightarrow y'\left(\frac{nx - ny + my + ny}{y(x-y)}\right) = \frac{mx + nx - mx + my}{x(x-y)}$

$\Rightarrow y'\left(\frac{nx+my}{y}\right) = \frac{nx+my}{x}$ $\therefore \frac{dy}{dx} = \frac{y}{x}$

Q05. Given $y = \sqrt{2x}^{\sqrt{2x}^{\sqrt{2x}\ldots\infty}}$ $\Rightarrow y = \left[\sqrt{2x}\right]^y$ $\Rightarrow \log y = y\log\sqrt{2x}$

i.e., $\log y = \frac{1}{2}y\log(2x)\ldots(i)$ $\Rightarrow \frac{1}{y}\times\frac{dy}{dx} = \frac{1}{2}\left[y\times\frac{1}{2x}(2) + \log(2x)\times\frac{dy}{dx}\right]$

$\Rightarrow \frac{1}{y}\times\frac{dy}{dx} = \frac{1}{2}\left[\frac{y}{x} + \log(2x)\times\frac{dy}{dx}\right]$ $\Rightarrow \frac{2}{y}\times\frac{dy}{dx} = \frac{y}{x} + \log(2x)\times\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left(\frac{2}{y} - \log(2x)\right) = \frac{y}{x}$ $\Rightarrow \frac{dy}{dx} = \frac{y^2}{x[2 - y\log(2x)]}$

By using (i), $\frac{dy}{dx} = \frac{y^2}{x[2 - 2\log y]}$ $\Rightarrow \frac{dy}{dx} = \frac{y^2}{2x[1-\log y]}$.

Q06. We've $y = \sin x^{\sin x^{\sin x\ldots\infty}}$ $\Rightarrow y = (\sin x)^y \ldots(i)$ $\Rightarrow y = e^{y\log\sin x}$

$\Rightarrow y' = e^{y\log\sin x}\left(y\times\frac{1}{\sin x}\times\cos x + y'\log\sin x\right)$ $\Rightarrow y' = (\sin x)^y\left(y\cot x + y'\log\sin x\right)$

$\Rightarrow y'\left(1 - (\sin x)^y\log\sin x\right) = y(\sin x)^y\cot x$ $\Rightarrow y' = \frac{y(\sin x)^y\cot x}{1 - (\sin x)^y\log\sin x}$

By using (i), we get : $\frac{dy}{dx} = \frac{y^2\cot x}{1 - y\log\sin x}$

Q07. Given $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \text{to } \infty}}}$ $\Rightarrow y = \sqrt{\cos x + y}$

$\Rightarrow y^2 - y = \cos x$ $\Rightarrow 2yy' - y' = -\sin x$ $\therefore \frac{dy}{dx} = \frac{\sin x}{1-2y}$

Q08. Given that $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \text{ to } \infty}}}$ $\Rightarrow y = \sqrt{\tan x + y}$

$\Rightarrow y^2 - y = \tan x$ $\Rightarrow 2yy' - y' = \sec^2 x$ $\therefore \frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$

Q09. We have $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \text{ to } \infty}}}$ $\Rightarrow y = \sqrt{\log x + y}$

$\Rightarrow y^2 - y = \log x \qquad \Rightarrow 2yy' - y' = \frac{1}{x} \qquad \therefore (2y-1)\frac{dy}{dx} = \frac{1}{x}$.

Based On Derivatives Of Functions Defined By Parameter

Q01. We have $x = \frac{2t}{1+t^2},\ y = \frac{1-t^2}{1+t^2}$ …(i) Differentiating w.r.t. t, we get :

$\frac{dx}{dt} = \frac{(1+t^2)2 - 2t(0+2t)}{(1+t^2)^2},\ \frac{dy}{dt} = \frac{(1+t^2)(0-2t)-(1-t^2)(0+2t)}{(1+t^2)^2}$

$\Rightarrow \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2},\ \frac{dy}{dt} = \frac{-4t}{(1+t^2)^2} \qquad \because \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dx}{dt}$

$\Rightarrow \frac{dy}{dx} = \frac{-4t}{(1+t^2)^2}\times\frac{(1+t^2)^2}{2(1-t^2)} \qquad \Rightarrow \frac{dy}{dx} = \frac{-2t}{1-t^2}$

By using (i), we get : $\frac{dy}{dx} = -\frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} \qquad \therefore \frac{dy}{dx} = -\frac{x}{y}$

Q02. We have $x = \sin^{-1}\left(\frac{2\theta}{1+\theta^2}\right),\ y = \cos^{-1}\left(\frac{1-\theta^2}{1+\theta^2}\right)$

Put $\theta = \tan\alpha \Rightarrow \alpha = \tan^{-1}\theta$ …(i)

So, $x = \sin^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right),\ y = \cos^{-1}\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right) \qquad \Rightarrow x = \sin^{-1}\sin 2\alpha,\ y = \cos^{-1}\cos 2\alpha$

$\Rightarrow x = 2\alpha,\ y = 2\alpha$ By (i), we get : $x = 2\tan^{-1}\theta,\ y = 2\tan^{-1}\theta$

$\Rightarrow y = x \qquad \therefore \frac{dy}{dx} = 1$

Q03. We have $\tan y = \frac{2t}{1-t^2},\ \sin x = \frac{2t}{1+t^2} \qquad \Rightarrow y = \tan^{-1}\frac{2t}{1-t^2},\ x = \sin^{-1}\frac{2t}{1+t^2}$

$\Rightarrow y = 2\tan^{-1}t,\ x = 2\tan^{-1}t \qquad \Rightarrow y = x \qquad \therefore \frac{dy}{dx} = 1$

Q04. Given $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Diff. w.r.t. θ, we get : $\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta.1),\ \frac{dy}{d\theta} = a(\cos\theta + \theta\sin\theta - \cos\theta.1)$

$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\theta,\ \frac{dy}{d\theta} = a\theta\sin\theta \qquad \because \frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right)$

$\therefore \frac{dy}{dx} = (a\theta\sin\theta)\left(\frac{1}{a\theta\cos\theta}\right) = \tan\theta$

Q05. **(a)** We have $x = ae^t(\sin t - \cos t),\ y = ae^t(\sin t + \cos t)$ On diff. w.r.t. t, we get :

$\frac{dx}{dt} = ae^t(\cos t + \sin t) + (\sin t - \cos t)ae^t,\ \frac{dy}{dt} = ae^t(\cos t - \sin t) + (\sin t + \cos t)ae^t$

$\Rightarrow \frac{dx}{dt} = 2ae^t\sin t,\ \frac{dy}{dt} = 2ae^t\cos t \qquad \because \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx} = 2ae^t\cos t \times \frac{1}{2ae^t\sin t} \qquad \therefore \frac{dy}{dx} = \cot t$

(b) Given that $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$

$\therefore \frac{dx}{dt} = ae^t(\cos t - \sin t) + ae^t(\sin t + \cos t)$ and $\frac{dy}{dt} = ae^t(\cos t + \sin t) + ae^t(\sin t - \cos t)$

$\Rightarrow \frac{dx}{dt} = 2ae^t \cos t$ and $\frac{dy}{dt} = 2ae^t \sin t$ $\qquad \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2ae^t \sin t}{2ae^t \cos t} = \frac{\sin t}{\cos t}$

$\Rightarrow \frac{dy}{dx} = \frac{ae^t(\sin t + \cos t) + ae^t(\sin t - \cos t)}{ae^t(\sin t + \cos t) - ae^t(\sin t - \cos t)}$ $\qquad$ So, $\frac{dy}{dx} = \frac{x+y}{x-y}$.

Q06. We have $x = \log t + \sin t, y = e^t + \cos t$ $\qquad \Rightarrow \frac{dx}{dt} = \frac{1}{t} + \cos t, \frac{dy}{dt} = e^t - \sin t$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = [e^t - \sin t] \times \frac{1}{\frac{1}{t} + \cos t}$ $\qquad \Rightarrow \frac{dy}{dx} = \frac{t[e^t - \sin t]}{1 + t\cos t}$

Q07. We have $x = \sqrt{1+t^2}$, $y = \sqrt{1-t^2}$ ...(i) $\qquad \Rightarrow x^2 - 1 = t^2$, $y^2 - 1 = -t^2$

$\Rightarrow x^2 + y^2 = 2$ $\qquad \Rightarrow 2x + 2yy' = 0$ $\qquad \Rightarrow y' = -x/y$

or, $\frac{dy}{dx} = -\sqrt{\frac{1+t^2}{1-t^2}}$. $\qquad$ [By using (i)

Q08. Given $x = \cos t,\ y = \log t$ $\qquad \Rightarrow \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \frac{1}{t}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \frac{-\operatorname{cosec} t}{t}$ ...(i)

Now diff. w.r.t. x both sides, $\frac{d^2y}{dx^2} = \frac{t\left(\operatorname{cosec} t \cot t \times \frac{dt}{dx}\right) - (-\operatorname{cosec} t).1 \times \frac{dt}{dx}}{t^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\operatorname{cosec} t(t\cot t + 1)}{t^2} \times \frac{dt}{dx}$ $\qquad \Rightarrow \frac{d^2y}{dx^2} = \frac{\operatorname{cosec} t(t\cot t + 1)}{t^2} \times \frac{1}{-\sin t}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\operatorname{cosec}^2 t(t\cot t + 1)}{t^2}$

Now $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\frac{\operatorname{cosec}^2 t(t\cot t + 1)}{t^2} + \left(\frac{-\operatorname{cosec} t}{t}\right)^2$

$\therefore$ Value of $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$ at $t = \frac{\pi}{2}$: $-\frac{\operatorname{cosec}^2(\pi/2)((\pi/2)\cot(\pi/2) + 1)}{(\pi/2)^2} + \left(\frac{-\operatorname{cosec}(\pi/2)}{\pi/2}\right)^2$

$\Rightarrow \quad = -\frac{(\pi/2)(0) + 1}{(\pi/2)^2} + \left(\frac{-1}{\pi/2}\right)^2 = -\frac{4}{\pi^2} + \frac{4}{\pi^2} = 0$.

Q09. a) We have $y = a\left\{\cos t + \log\tan\left(\frac{t}{2}\right)\right\}$, $x = a\sin t$

Diff. w.r.t. t both the sides, we get : $\frac{dy}{dt} = a\left\{-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \sec^2\left(\frac{t}{2}\right) \times \frac{1}{2}\right\}$, $\frac{dx}{dt} = a\cos t$

$\Rightarrow \frac{dy}{dt} = a\left\{-\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}\right\} = a\left\{-\sin t + \frac{1}{\sin t}\right\} = a\left\{\frac{1 - \sin^2 t}{\sin t}\right\} = a\cot t\cos t$

Now, $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = a\cot t\cos t \times \frac{1}{a\cos t} = \cot t$

Differentiating w.r.t. x both sides, $\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 t \times \frac{dt}{dx}$

i.e., $\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 t \times \frac{1}{a\cos t} = -\frac{(\sec t)(\operatorname{cosec}^2 t)}{a}$

b) We have $x = a\left\{\cos\theta + \log\tan\left(\frac{\theta}{2}\right)\right\}$, $y = a\sin\theta$

Diff. w.r.t. θ both the sides, we get : $\frac{dx}{d\theta} = a\left\{-\sin\theta + \frac{1}{\tan\left(\frac{\theta}{2}\right)} \times \sec^2\left(\frac{\theta}{2}\right) \times \frac{1}{2}\right\}$, $\frac{dy}{d\theta} = a\cos\theta$

$$\Rightarrow \frac{dx}{d\theta} = a\left\{-\sin\theta + \frac{1}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right\} = a\left\{-\sin\theta + \frac{1}{\sin\theta}\right\} = a\left\{\frac{1-\sin^2\theta}{\sin\theta}\right\} = a\cot\theta\cos\theta$$

Now, $\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right) = a\cos\theta \times \frac{1}{a\cos\theta\cot\theta} = \tan\theta$

Differentiating w.r.t. x both sides, $\frac{d^2y}{dx^2} = \sec^2\theta \times \frac{d\theta}{dx}$

i.e., $\frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{a\cot\theta\cos\theta} = \frac{\sec^3\theta\tan\theta}{a}$

So, $\left.\frac{d^2y}{dx^2}\right]_{at\ \theta=\pi/6} = \frac{\sec^3\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right)}{a} = \frac{\frac{8}{3\sqrt{3}} \times \frac{1}{\sqrt{3}}}{a} = \frac{8}{9a}$.

c) Proceed same as in Q09 (b) to get : $\frac{d^2y}{dx^2} = \frac{1}{a}\sec^3 t\tan t$. $\quad \therefore \left.\frac{d^2y}{dx^2}\right]_{at\ t=\pi/3} = \frac{8\sqrt{3}}{a}$.

Q10. Given $x = \cos\theta + \sin\theta,\ y = \sin\theta - \cos\theta$ $\quad \Rightarrow \frac{dx}{d\theta} = -\sin\theta + \cos\theta,\ \frac{dy}{d\theta} = \cos\theta + \sin\theta$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right) = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$ $\quad \Rightarrow \frac{dy}{dx} = \frac{1+\tan\theta}{1-\tan\theta}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(1-\tan\theta)\sec^2\theta \times \frac{d\theta}{dx} - (1+\tan\theta)(-\sec^2\theta) \times \frac{d\theta}{dx}}{(1-\tan\theta)^2} = \frac{2\sec^2\theta}{(1-\tan\theta)^2} \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2\sec^2\theta}{(1-\tan\theta)^2} \times \frac{1}{\cos\theta - \sin\theta}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{at\ \theta=\pi/3} = \frac{2\sec^2\frac{\pi}{3}}{\left(1-\tan\frac{\pi}{3}\right)^2} \times \frac{1}{\cos\frac{\pi}{3} - \sin\frac{\pi}{3}} = \frac{2(2)^2}{(1-\sqrt{3})^2} \times \frac{1}{\frac{1}{2} - \frac{\sqrt{3}}{2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{8}{(1-\sqrt{3})^2} \times \frac{2}{1-\sqrt{3}} = \frac{8}{2-\sqrt{3}} \times \frac{1}{1-\sqrt{3}} = \frac{8}{2-2\sqrt{3}-\sqrt{3}+3}$$

$\Rightarrow \quad = \frac{8}{5-3\sqrt{3}} \times \frac{5+3\sqrt{3}}{5+3\sqrt{3}} = \frac{8}{25-27} \times [5+3\sqrt{3}] = -4[5+3\sqrt{3}]$

Q11. We have $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x \ldots(i)$

$\Rightarrow y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow y = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\tan\frac{\theta}{2} = \frac{\theta}{2} \qquad \Rightarrow y = \frac{1}{2}\tan^{-1} x$ By (i).

Now we have $y = \frac{1}{2}\tan^{-1} x,\ z = \tan^{-1} x$ Diff. w.r.t. x both sides,

$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{1+x^2},\ \frac{dz}{dx} = \frac{1}{1+x^2} \qquad \therefore \frac{dy}{dz} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right) = \frac{1}{2} \times \frac{1}{1+x^2} \times [1+x^2] = \frac{1}{2}$

Q12. We have $y = \sin^{-1}(2x) + 2\sin^{-1}(\sqrt{1-4x^2})$ Put $2x = \sin\theta \Rightarrow \theta = \sin^{-1} 2x \ldots(i)$

$\Rightarrow y = \sin^{-1}(\sin\theta) + 2\sin^{-1}(\sqrt{1-\sin^2\theta}) \qquad \Rightarrow y = \theta + 2\sin^{-1}\cos\theta$ [By (i)

$\Rightarrow y = \sin^{-1} 2x + 2\sin^{-1}\sin\left(\frac{\pi}{2} - \theta\right) = \sin^{-1} 2x + \pi - 2\sin^{-1} 2x = \pi - \sin^{-1} 2x$

$\therefore \frac{dy}{dx} = -\frac{2}{\sqrt{1-4x^2}}$ Also, $z = \frac{1}{2}\cos^{-1} x \quad \frac{dz}{dx} = -\frac{1}{2} \times \frac{1}{\sqrt{1-x^2}}$

So, $\frac{dy}{dz} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right) = \left(-\frac{2}{\sqrt{1-4x^2}}\right)\left[-2\sqrt{1-x^2}\right] = 4\sqrt{\frac{1-x^2}{1-4x^2}}$

Q13. Given $y = \tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right)$ Put $ax = \tan\theta \Rightarrow \theta = \tan^{-1} ax \ldots(i)$

$\Rightarrow y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow y = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\tan\frac{\theta}{2} = \frac{\theta}{2} \qquad \Rightarrow y = \frac{1}{2}\tan^{-1} ax$ By (i)

So, $\frac{dy}{dx} = \frac{1}{2} \times \frac{a}{1+a^2x^2}$

Also, $z = \cot^{-1}(ax) \quad \Rightarrow \frac{dz}{dx} = -\frac{a}{1+a^2x^2}$

$\therefore \frac{dy}{dz} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right) = \left(\frac{1}{2} \times \frac{a}{1+a^2x^2}\right)\left(-\frac{(1+a^2x^2)}{a}\right) = -\frac{1}{2}$

Q14. We have $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ Put $x^2 = \cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1} x^2 \ldots(i)$

$$\Rightarrow y=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$\Rightarrow y=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)=\frac{\pi}{4}-\theta$$

By (i), $y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2 \quad \Rightarrow \frac{dy}{dx}=0-\frac{1}{2}\times\frac{-2x}{\sqrt{1-x^4}}=\frac{x}{\sqrt{1-x^4}}$

Also, $z=\sin^{-1}x^2 \quad \Rightarrow \frac{dz}{dx}=\frac{2x}{\sqrt{1-x^4}}$

$$\therefore \frac{dy}{dz}=\left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right)=\left(\frac{x}{\sqrt{1-x^4}}\right)\left(\frac{\sqrt{1-x^4}}{2x}\right)=\frac{1}{2}$$

Another Similar Sum : Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}x^2$.

Sol. Let $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ and $z=\cos^{-1}x^2 \Rightarrow x^2=\cos z$

$$\therefore y=\tan^{-1}\left(\frac{\sqrt{1+\cos z}-\sqrt{1-\cos z}}{\sqrt{1+\cos z}+\sqrt{1-\cos z}}\right)=\tan^{-1}\left(\frac{\sqrt{2}\cos\frac{z}{2}-\sqrt{2}\sin\frac{z}{2}}{\sqrt{2}\cos\frac{z}{2}+\sqrt{2}\sin\frac{z}{2}}\right)$$

$$\Rightarrow y=\tan^{-1}\left(\frac{\cos\frac{z}{2}-\sin\frac{z}{2}}{\cos\frac{z}{2}+\sin\frac{z}{2}}\right)=\tan^{-1}\left(\frac{1-\tan\frac{z}{2}}{1+\tan\frac{z}{2}\times 1}\right)=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{z}{2}\right)\right)=\frac{\pi}{4}-\frac{z}{2}$$

$$\therefore \frac{dy}{dx}=0-\frac{1}{2}\times\frac{d}{dz}(z)=-\frac{1}{2}.$$

Q15. We have $y=\tan^{-1}\left(\frac{x}{1+\sqrt{1+x^2}}\right)$ Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$ …(i)

$$\Rightarrow y=\tan^{-1}\left(\frac{\tan\theta}{1+\sqrt{1+\tan^2\theta}}\right)=\tan^{-1}\left(\frac{\tan\theta}{1+\sec\theta}\right) \quad \Rightarrow y=\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$\Rightarrow y=\tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)=\tan^{-1}\tan\frac{\theta}{2}=\frac{\theta}{2} \quad \Rightarrow y=\frac{1}{2}\tan^{-1}x \quad \text{By (i)}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\times\frac{1}{1+x^2}$$

Also $z=\sin^2\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$ Put $x=\cos\theta$ …(ii)

$$\Rightarrow z=\sin^2\left(2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)=\sin^2\left(2\tan^{-1}\sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\right)$$

$$\Rightarrow z=\sin^2\left(2\tan^{-1}\tan\frac{\theta}{2}\right)=\sin^2\theta=1-\cos^2\theta \quad \Rightarrow z=1-x^2 \quad \text{By (ii)}$$

$\therefore \frac{dz}{dx} = -2x$ So, $\frac{dy}{dz} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right) = \frac{1}{2} \times \frac{1}{1+x^2} \times \left(\frac{1}{-2x}\right)$

i.e., $\frac{dy}{dz} = -\frac{1}{4x[1+x^2]}$

Q16. We have $y = \tan^{-1}\left(\sqrt{1+x^2} - x\right)$ Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x \ldots$(i)

$\Rightarrow y = \tan^{-1}\left(\sqrt{1+\tan^2\theta} - \tan\theta\right)$ $\Rightarrow y = \tan^{-1}(\sec\theta - \tan\theta)$

$\Rightarrow y = \tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right)$ $\Rightarrow y = \tan^{-1}\left(\frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}\right)$

$\Rightarrow y = \tan^{-1}\left(\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{\pi}{4} - \frac{\theta}{2}$

By (i), $y = \frac{\pi}{4} - \frac{1}{2}\tan^{-1} x$ $\Rightarrow \frac{dy}{dx} = -\frac{1}{2} \times \frac{1}{1+x^2}$

Also, $z = \cot^{-1} x$ $\Rightarrow \frac{dz}{dx} = -\frac{1}{1+x^2}$

$\therefore \frac{dy}{dz} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right) = \left(-\frac{1}{2} \times \frac{1}{1+x^2}\right)\left(-(1+x^2)\right) = \frac{1}{2}$

Q17. We have $y = \sin^{-1}[2x\sqrt{1-x^2}]$ Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x \ldots$(i)

$\Rightarrow y = \sin^{-1}[2\sin\theta\sqrt{1-\sin^2\theta}]$ $\Rightarrow y = \sin^{-1}[\sin 2\theta] = 2\theta$

By (i), $y = 2\sin^{-1} x$ $\therefore \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

Also $z = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$ Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x \ldots$(ii)

$\Rightarrow z = \tan^{-1}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ $\Rightarrow z = \tan^{-1}\frac{\sin\theta}{\cos\theta} = \tan^{-1}\tan\theta = \theta$

By (ii), $z = \sin^{-1} x$ $\Rightarrow \frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$

So, $\frac{dy}{dz} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right) = \frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2} = 2$

Q18. Let $y = \frac{x}{\sin x}$ $\Rightarrow \frac{dy}{dx} = \frac{\sin x.1 - x\cos x}{\sin^2 x} = \frac{\sin x - x\cos x}{\sin^2 x}$

Also $z = \sin x$ $\Rightarrow \frac{dz}{dx} = \cos x$

$\therefore \frac{dy}{dz} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dz}\right) = \frac{\sin x - x\cos x}{\sin^2 x} \times \frac{1}{\cos x}$ $\Rightarrow \frac{dy}{dz} = \frac{\sin x - x\cos x}{\sin^2 x\cos x}$

Q19. Consider $x^2 + y^2 = t - \frac{1}{t}$ On squaring both the sides we get :

$x^4 + y^4 + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$ $\Rightarrow y^2 = -\frac{1}{x^2}$ $\left[\text{By using } x^4 + y^4 = t^2 + \frac{1}{t^2}\right.$

Differentiating w.r.t. x both the sides, $y^2 = -\frac{1}{x^2}$ $\Rightarrow 2y \times \frac{dy}{dx} = -\left(\frac{-2}{x^3}\right)$

$\Rightarrow 2y \times \frac{dy}{dx} = \frac{2}{x^3}$ $\therefore \frac{dy}{dx} = \frac{1}{yx^3}$

Q20. We have $x = \sin t$ $\Rightarrow \frac{dx}{dt} = \cos t$ And, $y = \sin mt$ $\Rightarrow \frac{dy}{dt} = m\cos mt$

$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \frac{m\cos mt}{\cos t}$ $\Rightarrow \cos t \frac{dy}{dx} = m\cos mt$

Now differentiating w.r.t. x both the sides, we get :

$\cos t \frac{d^2y}{dx^2} - \sin t \frac{dt}{dx}\frac{dy}{dx} = -m^2 \sin mt \frac{dt}{dx}$ $\Rightarrow \cos t \frac{d^2y}{dx^2} - \frac{\sin t}{\cos t}\frac{dy}{dx} = -m^2 \frac{\sin mt}{\cos t}$

$\Rightarrow \cos^2 t \frac{d^2y}{dx^2} - \sin t \frac{dy}{dx} = -m^2 \sin mt$ $\Rightarrow \cos^2 t \frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$

$\Rightarrow (1-\sin^2 t)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$ $\therefore (1-x^2)y_2 - xy_1 + m^2 y = 0$

Q21. We have $x = e^{\sin 2t}$ and $y = e^{\cos 2t}$ $\Rightarrow \frac{dx}{dt} = 2e^{\sin 2t}\cos 2t$ and $\frac{dy}{dt} = -2e^{\cos 2t}\sin 2t$

$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = -2e^{\cos 2t}\sin 2t \times \frac{1}{2e^{\sin 2t}\cos 2t}$ $\Rightarrow \frac{dy}{dx} = -\frac{e^{\cos 2t}}{e^{\sin 2t}} \times \frac{\sin 2t}{\cos 2t}$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x} \times \frac{\log x}{\log y} = -\frac{y\log x}{x\log y}$ $\left[\begin{array}{l}\because x = e^{\sin 2t} \Rightarrow \log x = \sin 2t \\ \text{and } y = e^{\cos 2t} \Rightarrow \log y = \cos 2t\end{array}\right.$

Q22. We have $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$ $\Rightarrow \log x = \log\sqrt{a^{\sin^{-1} t}}$ and $\log y = \log\sqrt{a^{\cos^{-1} t}}$

$\Rightarrow \log x = \frac{1}{2}\sin^{-1} t \log a$ and $\log y = \frac{1}{2}\cos^{-1} t \log a$

$\Rightarrow \frac{1}{x} \times \frac{dx}{dt} = \frac{1}{2}\log a \times \frac{1}{\sqrt{1-t^2}}$ and $\frac{1}{y} \times \frac{dy}{dt} = -\frac{1}{2}\log a \times \frac{1}{\sqrt{1-t^2}}$

$\Rightarrow \frac{dx}{dt} = \frac{1}{2}\log a \times \frac{x}{\sqrt{1-t^2}}$ and $\frac{dy}{dt} = -\frac{1}{2}\log a \times \frac{y}{\sqrt{1-t^2}}$

$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \left(-\frac{1}{2}\log a \times \frac{y}{\sqrt{1-t^2}}\right)\left(\frac{2\sqrt{1-t^2}}{x\log a}\right)$ $\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

Q23. We've $x = \sec\theta - \cos\theta$ $\Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$ $\Rightarrow \frac{dx}{d\theta} = [\sec\theta + \cos\theta]\tan\theta$

Also $y = \sec^4\theta - \cos^4\theta$ $\Rightarrow \frac{dy}{d\theta} = 4\sec^3\theta\sec\theta\tan\theta + 4\cos^3\theta\sin\theta$

$\Rightarrow \frac{dy}{d\theta} = 4\tan\theta[\sec^4\theta + \cos^4\theta]$ $\because \frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right)$

$\therefore \frac{dy}{dx} = \frac{4\tan\theta[\sec^4\theta + \cos^4\theta]}{[\sec\theta + \cos\theta]\tan\theta}$ $\Rightarrow [\sec\theta + \cos\theta]\frac{dy}{dx} = 4[\sec^4\theta + \cos^4\theta]$

Squaring both the sides, $[\sec\theta + \cos\theta]^2\left(\frac{dy}{dx}\right)^2 = 16[\sec^4\theta + \cos^4\theta]^2$

$$\left[\begin{array}{l}\because x=\sec\theta-\cos\theta \Rightarrow x^2=\sec^2\theta+\cos^2\theta-2 \Rightarrow x^2+4=\sec^2\theta+\cos^2\theta+2=(\sec\theta+\cos\theta)^2 \\ \text{and similarly, } y=\sec^4\theta-\cos^4\theta \Rightarrow y^2+4=(\sec^4\theta+\cos^4\theta)^2\end{array}\right]$$

$$\therefore (x^2+4)\left(\frac{dy}{dx}\right)^2=16(y^2+4).$$

Also try : If $x=\sec\theta-\cos\theta$ and $y=\sec^n\theta-\cos^n\theta$, show that $(x^2+4)\left(\frac{dy}{dx}\right)^2=n^2(y^2+4)$.

Q24. We have $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}$ $\Rightarrow \log x=3\log\sin t-\frac{1}{2}\log\cos 2t$

$$\Rightarrow \frac{1}{x}\times\frac{dx}{dt}=3\frac{\cos t}{\sin t}-\frac{1}{2}\frac{[-2\sin 2t]}{\cos 2t} \qquad \Rightarrow \frac{dx}{dt}=\frac{\sin^3 t}{\sqrt{\cos 2t}}[3\cot t+\tan 2t]$$

Similarly, $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$ $\Rightarrow \log y=3\log\cos t-\frac{1}{2}\log\cos 2t$

$$\Rightarrow \frac{1}{y}\times\frac{dy}{dt}=-3\tan t+\tan 2t \qquad \Rightarrow \frac{dy}{dt}=\frac{\cos^3 t}{\sqrt{\cos 2t}}[-3\tan t+\tan 2t]$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{\cos^3 t}{\sqrt{\cos 2t}}[-3\tan t+\tan 2t]\frac{\sqrt{\cos 2t}}{[3\cot t+\tan 2t]\sin^3 t}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos^3 t}{\sin^3 t}\left[\frac{-3\tan t+\tan 2t}{3\cot t+\tan 2t}\right] \qquad \Rightarrow \frac{dy}{dx}=\frac{\cos^2 t}{\sin^2 t}\left[\frac{-3\sin t\cos 2t+\sin 2t\cos t}{3\cos t\cos 2t+\sin 2t\sin t}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{-3\sin t\cos 2t\cos^2 t+2\sin t\cos t\cos^3 t}{3\sin^2 t\cos t\cos 2t+2\sin t\cos t\sin^3 t}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin t\cos t[-3\cos 2t\cos t+2\cos^3 t]}{\sin t\cos t[3\sin t\cos 2t+2\sin^3 t]} \qquad \Rightarrow \frac{dy}{dx}=\frac{-3(2\cos^2 t-1)\cos t+2\cos^3 t}{3\sin t(1-2\sin^2 t)+2\sin^3 t}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3\cos t-4\cos^3 t}{3\sin t-4\sin^3 t} \qquad \Rightarrow \frac{dy}{dx}=\frac{-(4\cos^3 t-3\cos t)}{\sin 3t}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\cos 3t}{\sin 3t} \qquad \Rightarrow \frac{dy}{dx}=-\cot 3t \qquad \therefore \frac{dy}{dx}+\cot 3t=0$$

Q25. Let $y=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$. Put $x=\cos\theta \Rightarrow \theta=\cos^{-1}x$...(i)

$$\therefore\ y=\tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \qquad \Rightarrow y=\tan^{-1}\tan\theta=\theta \ \Rightarrow y=\cos^{-1}x \ \text{[By (i)]}$$

On differentiating w.r.t. x both sides, we get : $\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$.

Also let $z=\cos^{-1}[2x\sqrt{1-x^2}]$. Put $x=\sin\theta \Rightarrow \theta=\sin^{-1}x$...(ii)

$$\therefore\ z=\cos^{-1}[2\sin\theta\sqrt{1-\sin^2\theta}]=\cos^{-1}[2\sin\theta\cos\theta] \qquad \Rightarrow z=\cos^{-1}[\sin 2\theta]=\frac{\pi}{2}-\sin^{-1}[\sin 2\theta]$$

$$\Rightarrow z=\frac{\pi}{2}-2\theta \qquad \Rightarrow z=\frac{\pi}{2}-2\sin^{-1}x \ \text{[By (ii)]}$$

On differentiating w.r.t. x both sides, we get : $\frac{dz}{dx}=0-2\times\frac{1}{\sqrt{1-x^2}}=-\frac{2}{\sqrt{1-x^2}}$.

Now, $\frac{dy}{dz}=\frac{dy}{dx}\times\frac{dx}{dz}=\left(-\frac{1}{\sqrt{1-x^2}}\right)\times\left(-\frac{\sqrt{1-x^2}}{2}\right)$ $\therefore \frac{dy}{dz}=\frac{1}{2}$.

NOTE : If we substitute $x=\cos\theta$ in $z=\cos^{-1}[2x\sqrt{1-x^2}]$ then the answer will be $-\frac{1}{2}$, which will be correct as well (as mentioned in the **CBSE Marking Scheme**).

Q26. Let $y=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$. Put $x=\sin\theta \Rightarrow \theta=\sin^{-1}x$...(i)

$\therefore\ y=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$ $\Rightarrow y=\tan^{-1}\tan\theta=\theta$ $\Rightarrow y=\sin^{-1}x$ [By (i)

On differentiating w.r.t. x both sides, we get : $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$.

Also let $z=\sin^{-1}[2x\sqrt{1-x^2}]$. Put $x=\sin\theta \Rightarrow \theta=\sin^{-1}x$...(ii)

$\therefore\ z=\sin^{-1}[2\sin\theta\sqrt{1-\sin^2\theta}]=\sin^{-1}[2\sin\theta\cos\theta]$ $\Rightarrow z=\sin^{-1}[\sin 2\theta]$

$\Rightarrow z=2\theta$ $\Rightarrow z=2\sin^{-1}x$ [By (ii)

On differentiating w.r.t. x both sides, we get : $\frac{dz}{dx}=2\times\frac{1}{\sqrt{1-x^2}}$.

Now, $\frac{dy}{dz}=\frac{dy}{dx}\times\frac{dx}{dz}=\frac{1}{\sqrt{1-x^2}}\times\frac{\sqrt{1-x^2}}{2}$ $\therefore\ \frac{dy}{dz}=\frac{1}{2}$.

NOTE : If we substitute $x=\cos\theta$ in $z=\sin^{-1}[2x\sqrt{1-x^2}]$ then the answer will be $-\frac{1}{2}$, which will be correct as well (as mentioned in the **CBSE Marking Scheme**).

Q27. Let $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$. Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$...(i)

$\therefore y=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)=\tan^{-1}\tan\frac{\theta}{2}=\frac{\theta}{2}$ $\Rightarrow y=\frac{1}{2}\tan^{-1}x$ [By (i)

On differentiating w.r.t. x both sides, we get : $\frac{dy}{dx}=\frac{1}{2}\times\frac{1}{1+x^2}$.

Also let $z=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$. Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$...(ii)

$\therefore z=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}\sin 2\theta=2\theta$ $\Rightarrow z=2\tan^{-1}x$ [By (ii)

On differentiating w.r.t. x both sides, we get : $\frac{dz}{dx}=2\times\frac{1}{1+x^2}$.

Now, $\frac{dy}{dz}=\frac{dy}{dx}\times\frac{dx}{dz}=\frac{dy}{dx}=\left(\frac{1}{2}\times\frac{1}{1+x^2}\right)\left(\frac{1+x^2}{2}\right)$ $\therefore \frac{dy}{dz}=\frac{1}{4}$.

NOTE : If we substitute $x=\cot\theta$ then the answer will be $-\frac{1}{4}$, which will be correct as well (as mentioned in the **CBSE Marking Scheme**).

Q28. **(a)** Given $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$

Diff. w.r.t. t, we get : $\frac{dx}{dt}=a(-\sin t+t\cos t+\sin t.1)$, $\frac{dy}{dt}=a(\cos t+t\sin t-\cos t.1)$

$\Rightarrow \frac{dx}{dt}=at\cos t$, $\frac{dy}{dt}=at\sin t$ $\qquad \because \frac{dy}{dx}=\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)$

$\therefore \frac{dy}{dx}=(at\sin t)\left(\frac{1}{at\cos t}\right)=\tan t$ $\qquad \Rightarrow \frac{d^2y}{dx^2}=\sec^2 t\times\frac{dt}{dx}$ $\qquad \Rightarrow \frac{d^2y}{dx^2}=\sec^2 t\times\frac{1}{at\cos t}$

$\therefore \left.\frac{d^2y}{dx^2}\right]_{at\ t=\frac{\pi}{4}}=\sec^3\frac{\pi}{4}\times\frac{1}{a\times\frac{\pi}{4}}=\frac{8\sqrt{2}}{a\pi}$.

(b) Given $x=a(\cos 2t+2t\sin 2t)$ and $y=a(\sin 2t-2t\cos 2t)$

$\therefore \frac{dx}{dt}=a(-2\sin 2t+4t\cos 2t+2\sin 2t)=4at\cos 2t$ and

$\frac{dy}{dt}=a(2\cos 2t+4t\sin 2t-2\cos 2t)=4at\sin 2t$

So, $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{4at\sin 2t}{4at\cos 2t}=\tan 2t$ $\qquad \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=2\sec^2 2t\times\frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=2\sec^2 2t\times\frac{1}{4at\cos 2t}=\frac{\sec^3 2t}{2at}$.

Q29. We have $x=\sin t\sqrt{\cos 2t}$ $\qquad \Rightarrow \frac{dx}{dt}=\sin t\times\frac{-2\sin 2t}{2\sqrt{\cos 2t}}+\sqrt{\cos 2t}\times\cos t$

$\Rightarrow \frac{dx}{dt}=\frac{\cos 2t\cos t-\sin t\sin 2t}{\sqrt{\cos 2t}}=\frac{\cos 3t}{\sqrt{\cos 2t}}$

And, $y=\cos t\sqrt{\sin 2t}$ $\qquad \Rightarrow \frac{dy}{dt}=\cos t\times\frac{2\cos 2t}{2\sqrt{\sin 2t}}-\sqrt{\sin 2t}\times\sin t$

$\Rightarrow \frac{dy}{dt}=\frac{\cos 2t\cos t-\sin t\sin 2t}{\sqrt{\sin 2t}}=\frac{\cos 3t}{\sqrt{\sin 2t}}$ $\qquad \therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{\cos 3t}{\sqrt{\sin 2t}}\times\frac{\sqrt{\cos 2t}}{\cos 3t}$

So, $\left.\frac{dy}{dx}\right]_{at\ t=\frac{\pi}{8}}=\frac{\sqrt{\cos\left(2\times\frac{\pi}{8}\right)}}{\sqrt{\sin\left(2\times\frac{\pi}{8}\right)}}=\sqrt{\frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}}=1$

Q30. Given $x=2\cos\theta-\cos 2\theta$ and, $y=2\sin\theta-\sin 2\theta$

$\Rightarrow \frac{dx}{d\theta}=-2\sin\theta+2\sin 2\theta$ and, $\frac{dy}{d\theta}=2\cos\theta-2\cos 2\theta$

$\therefore \frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{\cos\theta-\cos 2\theta}{\sin 2\theta-\sin\theta}$ $\qquad \Rightarrow \frac{dy}{dx}=\frac{-2\sin\left(\frac{3\theta}{2}\right)\sin\left(-\frac{\theta}{2}\right)}{2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\left(\frac{3\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)} \qquad \therefore \frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right) \qquad [\because \sin(-x) = -\sin x$$

Q31. Given $x = 2\cos\theta - \cos 2\theta$ and, $y = 2\sin\theta - \sin 2\theta$

$$\Rightarrow \frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta \text{ and, } \frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta} \qquad \Rightarrow \frac{dy}{dx} = \frac{-2\sin\left(\frac{3\theta}{2}\right)\sin\left(-\frac{\theta}{2}\right)}{2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right) \qquad \Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}\sec^2\left(\frac{3\theta}{2}\right) \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{3}{4}\sec^2\left(\frac{3\theta}{2}\right) \times \frac{1}{\sin\theta - \sin 2\theta}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right]_{\text{at } \theta=\frac{\pi}{2}} = -\frac{3}{4}\sec^2\left(\frac{3}{2} \times \frac{\pi}{2}\right) \times \frac{1}{\sin\frac{\pi}{2} - \sin\left(2 \times \frac{\pi}{2}\right)}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right]_{\text{at } \theta=\frac{\pi}{2}} = -\frac{3}{4}\sec^2\left(\frac{3\pi}{4}\right) \times \frac{1}{1 - \sin\pi} \qquad \Rightarrow \left.\frac{d^2y}{dx^2}\right]_{\text{at } \theta=\frac{\pi}{2}} = -\frac{3}{4}\left(\sqrt{2}\right)^2 \times \frac{1}{1-0} = -\frac{3}{2}$$

$\therefore$ Value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$ is $-\frac{3}{2}$

Q32. **(a)** We have $x = a\sin 2t(1 + \cos 2t) \Rightarrow \frac{dx}{dt} = 2a\cos 2t(1 + \cos 2t) - 2a\sin^2 2t$

And, $y = b\cos 2t(1 - \cos 2t) \Rightarrow \frac{dy}{dt} = 2b\cos 2t\sin 2t - 2b\sin 2t(1 - \cos 2t)$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2b\cos 2t\sin 2t - 2b\sin 2t(1 - \cos 2t)}{2a\cos 2t(1 + \cos 2t) - 2a\sin^2 2t} = \frac{b[\cos 2t\sin 2t - \sin 2t(1 - \cos 2t)]}{a[\cos 2t(1 + \cos 2t) - \sin^2 2t]}$$

Since we know that at $t = \frac{\pi}{4}$, $\sin 2t = 1$ and $\cos 2t = 0$,

So, $\frac{dy}{dx}\left(t = \frac{\pi}{4}\right) = \frac{b[0.1 - 1(1-0)]}{a[0(1+0) - 1^2]} = \frac{b}{a}$.

(b) We have $x = \alpha\sin 2t(1 + \cos 2t)$ and $y = \beta\cos 2t(1 - \cos 2t)$

$$\Rightarrow \frac{dx}{dt} = \alpha\left[-2\sin 2t\sin 2t + 2\cos 2t(1 + \cos 2t)\right] \text{ and } \frac{dy}{dt} = \beta\left[2\sin 2t\cos 2t - 2\sin 2t(1 - \cos 2t)\right]$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\beta\left[2\sin 2t\cos 2t - 2\sin 2t(1 - \cos 2t)\right]}{\alpha\left[-2\sin 2t\sin 2t + 2\cos 2t(1 + \cos 2t)\right]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\beta\left[\sin 4t - 2\sin 2t + 2\sin 2t\cos 2t\right]}{\alpha\left[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t\right]} = \frac{\beta\left[\sin 4t - 2\sin 2t + \sin 4t\right]}{\alpha\left[2\cos 4t + 2\cos 2t\right]} = \frac{\beta\left[2\sin 4t - 2\sin 2t\right]}{\alpha\left[2\cos 4t + 2\cos 2t\right]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\beta\left[\sin 4t - \sin 2t\right]}{\alpha\left[\cos 4t + \cos 2t\right]} = \frac{\beta\left[2\cos 3t\sin t\right]}{\alpha\left[2\cos 3t\cos t\right]} \qquad \therefore \frac{dy}{dx} = \frac{\beta}{\alpha}\tan t.$$

(c) We have $x = a\sin 2t(1+\cos 2t) \Rightarrow \frac{dx}{dt} = 2a\cos 2t(1+\cos 2t) - 2a\sin^2 2t$

And, $y = b\cos 2t(1-\cos 2t) \Rightarrow \frac{dy}{dt} = 2b\cos 2t\sin 2t - 2b\sin 2t(1-\cos 2t)$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{2b\cos 2t\sin 2t - 2b\sin 2t(1-\cos 2t)}{2a\cos 2t(1+\cos 2t) - 2a\sin^2 2t} = \frac{b[\cos 2t\sin 2t - \sin 2t(1-\cos 2t)]}{a[\cos 2t(1+\cos 2t) - \sin^2 2t]}$$

Since we know that at $t = \frac{\pi}{4}$, $\sin 2t = 1$ and $\cos 2t = 0$,

So, $\frac{dy}{dx}\left(t=\frac{\pi}{4}\right) = \frac{b[0.1-1(1-0)]}{a[0(1+0)-1^2]} = \frac{b}{a}$.

Also we know that at $t = \frac{\pi}{3}$, $\sin 2t = \frac{\sqrt{3}}{2}$ and $\cos 2t = -\frac{1}{2}$,

$$\text{So, } \frac{dy}{dx}\left(t=\frac{\pi}{3}\right) = \frac{b\left[\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}\right)\right]}{a\left[\left(-\frac{1}{2}\right)\left(1-\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)^2\right]} = \frac{b\left[-\frac{\sqrt{3}}{4} - \frac{3\sqrt{3}}{4}\right]}{a\left[-\frac{1}{4} - \frac{3}{4}\right]} = \frac{b\left[-\sqrt{3}\right]}{a[-1]} = \sqrt{3}\left(\frac{b}{a}\right).$$

Q33. Given $x = \cos t(3-2\cos^2 t) = 3\cos t - 2\cos^3 t$

$\Rightarrow \frac{dx}{dt} = 6\cos^2 t\sin t - 3\sin t = 3\sin t(2\cos^2 t - 1)$

And, $y = \sin t(3-2\sin^2 t) = 3\sin t - 2\sin^3 t \Rightarrow \frac{dy}{dt} = 3\cos t - 6\sin^2 t\cos t = 3\cos t(1-2\sin^2 t)$

$\therefore \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{3\cos t(1-2\sin^2 t)}{3\sin t(2\cos^2 t - 1)} = \cot t$ So, $\left.\frac{dy}{dx}\right]_{at\ t=\pi/4} = \cot(\pi/4) = 1$.

Q34. We have $x = a\sin pt$ and, $y = b\cos pt$ $\Rightarrow \frac{dx}{dt} = ap\cos pt$ and, $\frac{dy}{dt} = -bp\sin pt$

$\because \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$ $\Rightarrow \frac{dy}{dx} = -\frac{b\sin pt}{a\cos pt} = -\frac{b}{a}\tan pt$

$\therefore \frac{d^2y}{dx^2} = -\frac{bp}{a}\sec^2 pt\times\frac{dt}{dx} = -\frac{bp}{a}\sec^2 pt\times\frac{1}{ap\cos pt} = -\frac{b}{a^2}\sec^3 pt$

Therefore, $\left.\frac{d^2y}{dx^2}\right]_{at\ t=0} = -\frac{b}{a^2}\sec^3 0 = -\frac{b}{a^2}$.

Q35. Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ and $z = \sqrt{1-x^2}$ $\Rightarrow y = \cos^{-1}\left(2x^2-1\right)$ and $z = \sqrt{1-x^2}$

Replace $x = \cos\theta$ in y, we get : $y = \cos^{-1}\left(2\cos^2\theta - 1\right)$ and $z = \sqrt{1-x^2}$

$\Rightarrow y = \cos^{-1}\cos 2\theta = 2\theta = 2\cos^{-1}x$ and $z = \sqrt{1-x^2}$

Now differentiating both y and z w. r. t. x, $\frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$ and $\frac{dz}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dz} = \frac{dy}{dx}\times\frac{dx}{dz} = -\frac{2}{\sqrt{1-x^2}}\times\left(-\frac{\sqrt{1-x^2}}{x}\right) = \frac{2}{x}$ $\Rightarrow \left.\frac{dy}{dz}\right]_{at\ x=1/2} = \frac{2}{1/2} = 4$.

Q36. Here $x = a\left(t+\frac{1}{t}\right)$ and $y = a\left(t-\frac{1}{t}\right)$ $\Rightarrow \frac{dx}{dt} = a\left(1-\frac{1}{t^2}\right)$ and $\frac{dy}{dt} = a\left(1+\frac{1}{t^2}\right)$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{a\left(1+\frac{1}{t^2}\right)}{a\left(1-\frac{1}{t^2}\right)}=\frac{at\left(1+\frac{1}{t^2}\right)}{at\left(1-\frac{1}{t^2}\right)}=\frac{a\left(t+\frac{1}{t}\right)}{a\left(t-\frac{1}{t}\right)}=\frac{x}{y}.$$

Based On Derivatives Of Miscellaneous Functions

Q01. Given $y=\log(1-\cos x)$ $\Rightarrow y=\log\left(2\sin^2\frac{x}{2}\right)$ $\Rightarrow y=\log 2+2\log\left(\sin\frac{x}{2}\right)$

$\Rightarrow \frac{dy}{dx}=2\times\frac{1}{\sin\frac{x}{2}}\times\frac{d}{dx}\left(\sin\frac{x}{2}\right)$ $\Rightarrow \frac{dy}{dx}=2\times\frac{1}{\sin\frac{x}{2}}\times\cos\frac{x}{2}\times\frac{1}{2}$ $\Rightarrow \frac{dy}{dx}=\cot\frac{x}{2}$

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$ $\Rightarrow \frac{d^3y}{dx^3}=-\frac{1}{2}\left(2\operatorname{cosec}\frac{x}{2}\left[-\operatorname{cosec}\frac{x}{2}\cot\frac{x}{2}\times\frac{1}{2}\right]\right)$

$\therefore \frac{d^3y}{dx^3}=\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\cot\frac{x}{2}.$

Q02. We have $y=x^{x^{x\ldots\infty}}$ $\Rightarrow y=x^y$ $\Rightarrow y=e^{\log x^y}$ $\Rightarrow y=e^{y\log x}$

$\Rightarrow \frac{dy}{dx}=e^{y\log x}\left(y\times\frac{1}{x}+\log x\times\frac{dy}{dx}\right)$ $\Rightarrow \frac{dy}{dx}=x^y\left(\frac{y}{x}+\log x\times\frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx}(1-y\log x)=\frac{y^2}{x}$ $\therefore \frac{dy}{dx}=\frac{y^2}{x[1-y\log x]}$

Q03. We have $y=e^{x+e^{x+e^{x}+\ldots+\infty}}$ $\Rightarrow y=e^{x+y}$ Now take logarithm on both the sides

$\log y=\log e^{x+y}$ $\Rightarrow \log y=(x+y)\log e$ $[\because \log_e e=1$

$\therefore \log y=(x+y)$ $\Rightarrow \frac{1}{y}\times\frac{dy}{dx}=1+\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y}-1\right)\times\frac{dy}{dx}=1$ $\therefore \frac{dy}{dx}=\frac{y}{1-y}$

Q04. **(a)** We have $x=(a+bx)e^{y/x}$ $\Rightarrow \log x=\log[(a+bx)e^{y/x}]$

$\Rightarrow \log x=\log(a+bx)+\log e^{y/x}$ $\Rightarrow \log x-\log(a+bx)=\frac{y}{x}\log e$

or, $y=x\log\left(\frac{x}{a+bx}\right)$ $\Rightarrow y=x\ [\log x-\log(a+bx)]\ldots(A)$

$\Rightarrow \frac{dy}{dx}=x\left[\frac{1}{x}-\frac{b}{a+bx}\right]+[\log x-\log(a+bx)].1$ [By (A), $\frac{y}{x}=\log x-\log(a+bx)$]

So $\frac{dy}{dx}=\frac{a}{a+bx}+\frac{y}{x}$ ...(i) $\Rightarrow xy'=\frac{ax}{a+bx}+y$

$\Rightarrow xy''+y'=\frac{(a+bx).a-ax(b)}{(a+bx)^2}+y'$ $\Rightarrow xy''=\frac{a^2}{(a+bx)^2}=\left(y'-\frac{y}{x}\right)^2$ [By (i), $y'-\frac{y}{x}=\frac{a}{a+bx}$]

That is, $xy''=\frac{(xy'-y)^2}{x^2}$ $\Rightarrow x^3y''=(xy'-y)^2$ $\therefore\ x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2.$

(b) Given $y=\log\left(\frac{x}{a+bx}\right)^x$ or, $y=x\log\left(\frac{x}{a+bx}\right)$

$\Rightarrow y = x\,[\log x - \log(a+bx)]$ $\qquad\Rightarrow \frac{y}{x} = [\log x - \log(a+bx)]$

$\Rightarrow \frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{b}{a+bx} = \frac{a}{x(a+bx)}$ $\qquad\Rightarrow x\frac{dy}{dx} - y = \frac{ax}{a+bx}\ldots(i)$

Diff. w. r. t. x again, $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax(0+b)}{(a+bx)^2} = \frac{a^2}{(a+bx)^2}$

$\Rightarrow x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(\frac{ax}{a+bx}\right)$ $\qquad\therefore\ x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$ $\qquad$[By (i).

(c) We have $x = (ax+b)e^{y/x}$ $\qquad\Rightarrow \log x = \log[(ax+b)e^{y/x}]$

$\Rightarrow \log x = \log(ax+b) + \log e^{y/x}$ $\qquad\Rightarrow \log x - \log(ax+b) = \frac{y}{x}\log e$

or, $y = x\log\left(\frac{x}{ax+b}\right)$ $\qquad\Rightarrow y = x\,[\log x - \log(ax+b)]\ldots(A)$

$\Rightarrow \frac{dy}{dx} = x\left[\frac{1}{x} - \frac{a}{ax+b}\right] + [\log x - \log(ax+b)].1$ $\qquad$[By (A), $\frac{y}{x} = \log x - \log(ax+b)$]

So $\frac{dy}{dx} = \frac{b}{ax+b} + \frac{y}{x}$ ...(i) $\qquad\Rightarrow xy' = \frac{bx}{ax+b} + y$

$\Rightarrow xy'' + y' = \frac{(ax+b).b - bx(a)}{(ax+b)^2} + y'$ $\qquad\Rightarrow xy'' = \frac{b^2}{(ax+b)^2} = \left(y' - \frac{y}{x}\right)^2$ $\qquad$[By (i), $y' - \frac{y}{x} = \frac{b}{ax+b}$]

That is, $xy'' = \frac{(xy'-y)^2}{x^2}$ $\qquad\Rightarrow x^3y'' = (xy'-y)^2$ $\qquad\therefore\ x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

Q05. We have $f(x) = \frac{\log x}{x}$ $\qquad\Rightarrow f'(x) = \frac{x\times\frac{1}{x} - \log x.1}{x^2} = \frac{1-\log x}{x^2}$

$\Rightarrow f''(x) = \frac{x^2\left(0 - \frac{1}{x}\right) - (1-\log x)2x}{x^4}$ $\qquad\Rightarrow f''(x) = \frac{-x - (1-\log x)2x}{x^4} = -\left(\frac{1+2-2\log x}{x^3}\right)$

$\therefore f''(x) = \frac{2\log x - 3}{x^3}$

Q06. We have $y = \log(1-\sin x)$ $\qquad\Rightarrow \frac{dy}{dx} = \frac{-\cos x}{1-\sin x} = \frac{-\left[\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right]}{\left[\cos\frac{x}{2} - \sin\frac{x}{2}\right]^2} = -\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$\therefore \frac{d^2y}{dx^2} = -\frac{1}{2}\sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right)$ and, $\frac{d^3y}{dx^3} = -\frac{1}{2}\times 2\sec\left(\frac{\pi}{4} + \frac{x}{2}\right)\sec\left(\frac{\pi}{4} + \frac{x}{2}\right)\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\times\frac{1}{2}$

i.e., $\frac{d^3y}{dx^3} = -\sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right)\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\times\frac{1}{2}$

$\therefore y_3 + y_2y_1 = -\sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right)\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\times\frac{1}{2} + \left[-\frac{1}{2}\sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]\left[-\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = 0$

Q07. Given $\log y = \tan^{-1}x \Rightarrow y = e^{\tan^{-1}x}$ $\qquad\Rightarrow y' = e^{\tan^{-1}x}\times\frac{1}{1+x^2}$ $\qquad\Rightarrow (1+x^2)y' = e^{\tan^{-1}x}$

$\Rightarrow (1+x^2)y'' + y'(0+2x) = e^{\tan^{-1}x} \times \frac{1}{1+x^2}$ $\quad\Rightarrow (1+x^2)^2 y'' + 2x(1+x^2)y' = y$

$\Rightarrow (1+x^2)^2 y'' + 2x(1+x^2)y' - y = 0$

Q08. We have $y = \left\{\log[x+\sqrt{a^2+x^2}]\right\}^2$

$$\Rightarrow \frac{dy}{dx} = 2\log[x+\sqrt{a^2+x^2}]\left[\frac{1}{x+\sqrt{a^2+x^2}} \times \left(1+\frac{2x}{2\sqrt{x^2+a^2}}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = 2\log[x+\sqrt{a^2+x^2}]\left[\frac{1}{x+\sqrt{a^2+x^2}} \times \left(\frac{x+\sqrt{a^2+x^2}}{\sqrt{x^2+a^2}}\right)\right]$$

$\Rightarrow \frac{dy}{dx} = 2\log[x+\sqrt{a^2+x^2}] \times \frac{1}{\sqrt{x^2+a^2}}$ $\quad\Rightarrow \sqrt{x^2+a^2} \times \frac{dy}{dx} = 2\log[x+\sqrt{a^2+x^2}]$

$$\Rightarrow \sqrt{x^2+a^2} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{2x}{2\sqrt{x^2+a^2}} = \frac{2}{x+\sqrt{a^2+x^2}} \times \left[1+\frac{2x}{2\sqrt{a^2+x^2}}\right]$$

$$\Rightarrow \sqrt{x^2+a^2} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{x}{\sqrt{x^2+a^2}} = \frac{2}{\sqrt{a^2+x^2}}$$

$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$ $\quad$ i.e., $(x^2+a^2)y_2 + xy_1 = 2$.

Q09. We have $y = \left\{\log[x+\sqrt{1+x^2}]\right\}^2$

$$\Rightarrow \frac{dy}{dx} = 2\log[x+\sqrt{1+x^2}]\left[\frac{1}{x+\sqrt{1+x^2}} \times \left(1+\frac{2x}{2\sqrt{x^2+1}}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = 2\log[x+\sqrt{1+x^2}]\left[\frac{1}{x+\sqrt{1+x^2}} \times \left(\frac{x+\sqrt{1+x^2}}{\sqrt{x^2+1}}\right)\right]$$

$\Rightarrow \frac{dy}{dx} = 2\log[x+\sqrt{1+x^2}] \times \frac{1}{\sqrt{x^2+1}}$ $\quad\Rightarrow \sqrt{x^2+1} \times \frac{dy}{dx} = 2\log[x+\sqrt{1+x^2}]$

$$\Rightarrow \sqrt{x^2+1} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{2x}{2\sqrt{x^2+1}} = \frac{2}{x+\sqrt{1+x^2}} \times \left[1+\frac{2x}{2\sqrt{1+x^2}}\right]$$

$$\Rightarrow \sqrt{x^2+1} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{x}{\sqrt{x^2+1}} = \frac{2}{\sqrt{1+x^2}}$$

$\Rightarrow (x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$ $\quad$ i.e., $(x^2+1)y_2 + xy_1 - 2 = 0$

Q10. We have $x^y = e^{y-x}$ $\quad\Rightarrow \log x^y = \log e^{y-x}$ $\quad\Rightarrow y\log x = (y-x)\log e$

$\Rightarrow y\log x = y - x$ $\quad\Rightarrow y(1-\log x) = x$ $\quad\Rightarrow y = \frac{x}{(1-\log x)}$

$\Rightarrow \frac{dy}{dx} = \frac{(1-\log x).1 - x\left(0-\frac{1}{x}\right)}{(1-\log x)^2}$ $\quad\therefore \frac{dy}{dx} = \frac{2-\log x}{(1-\log x)^2}$.

Q11. Given $x^y = e^{x-y}$ $\quad\Rightarrow \log x^y = \log e^{x-y}$ $\quad\Rightarrow y\log x = (x-y)\log e$

$\Rightarrow y\log x=(x-y)$ $\quad\Rightarrow y(1+\log x)=x$ $\quad\Rightarrow y=\dfrac{x}{(1+\log x)}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{(1+\log x).1-x\left[0+\dfrac{1}{x}\right]}{(1+\log x)^2}$ $\quad\Rightarrow \dfrac{dy}{dx}=\dfrac{\log x}{(1+\log x)^2}$

Q12. We have $x^y=e^{x+y}$ $\quad\Rightarrow y\log x=(x+y)\log e$ $\quad\Rightarrow y\log x=(x+y)$

$\Rightarrow y(\log x-1)=x$ $\quad\Rightarrow y=\dfrac{x}{(\log x-1)}$ $\quad\Rightarrow \dfrac{dy}{dx}=\dfrac{(\log x-1).1-x\left(\dfrac{1}{x}-0\right)}{(\log x-1)^2}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{\log x-1-1}{(\log x-1)^2}$ $\quad\therefore \dfrac{dy}{dx}=\dfrac{\log x-2}{(\log x-1)^2}$

Q13. We have $x^\alpha y^\beta=(x+y)^{\alpha+\beta}$ $\quad\Rightarrow \log(x^\alpha y^\beta)=\log(x+y)^{\alpha+\beta}$

$\Rightarrow \alpha\log x+\beta\log y=(\alpha+\beta)\log(x+y)$ Differentiating w.r.t. x both sides, we get :

$\Rightarrow \alpha\times\dfrac{1}{x}+\beta\times\dfrac{1}{y}\times y'=(\alpha+\beta)\times\dfrac{1}{x+y}\times(1+y')$

$\Rightarrow \dfrac{\alpha}{x}+\dfrac{\beta}{y}\times y'=\dfrac{\alpha+\beta}{x+y}+y'\dfrac{\alpha+\beta}{x+y}$ $\quad\Rightarrow y'\left(\dfrac{\beta}{y}-\dfrac{\alpha+\beta}{x+y}\right)=\dfrac{\alpha+\beta}{x+y}-\dfrac{\alpha}{x}$

$\Rightarrow y'\left(\dfrac{\beta x+\beta y-\alpha y-\beta y}{y(x+y)}\right)=\dfrac{\alpha x+\beta x-\alpha x-\alpha y}{x(x+y)}$ $\quad\Rightarrow y'\left(\dfrac{\beta x-\alpha y}{y}\right)=\dfrac{\beta x-\alpha y}{x}$ $\quad\therefore \dfrac{dy}{dx}=\dfrac{y}{x}$

Also, $\dfrac{d^2y}{dx^2}=y\left(-\dfrac{1}{x^2}\right)+\dfrac{1}{x}\times\dfrac{dy}{dx}=-\dfrac{y}{x^2}+\dfrac{1}{x}\times\dfrac{y}{x}=0$

Q14. Here $x^{7/3}.\,y^{2/3}=(x+y)^3$ $\quad\Rightarrow \log(x^{7/3}y^{2/3})=\log(x+y)^3$

$\Rightarrow \dfrac{7}{3}\log x+\dfrac{2}{3}\log y=3\log(x+y)$ Differentiating w.r.t. x both sides, we get :

$\Rightarrow \dfrac{7}{3}\times\dfrac{1}{x}+\dfrac{2}{3}\times\dfrac{1}{y}\times y'=3\times\dfrac{1}{x+y}\times(1+y')$

$\Rightarrow \dfrac{7}{3x}+\dfrac{2}{3y}\times y'=\dfrac{3}{x+y}+y'\dfrac{3}{x+y}$ $\quad\Rightarrow y'\left(\dfrac{2}{3y}-\dfrac{3}{x+y}\right)=\dfrac{3}{x+y}-\dfrac{7}{3x}$

$\Rightarrow y'\left(\dfrac{2x+2y-9y}{3y(x+y)}\right)=\dfrac{9x-7x-7y}{3x(x+y)}$ $\quad\Rightarrow y'\left(\dfrac{2x-7y}{y}\right)=\dfrac{2x-7y}{x}$ $\quad\therefore \dfrac{dy}{dx}=\dfrac{y}{x}$

Q15. Same as Q13

Q16. We've $y=\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}-\log[\sqrt{1-x^2}]$ $\quad\Rightarrow y=\left(\dfrac{x}{\sqrt{1-x^2}}\right)\cos^{-1}x-\dfrac{1}{2}\log[1-x^2]$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{x}{\sqrt{1-x^2}}\left(-\dfrac{1}{\sqrt{1-x^2}}\right)+\cos^{-1}x\left(\dfrac{\sqrt{1-x^2}\times1-x\times\dfrac{(-2x)}{2\sqrt{1-x^2}}}{[\sqrt{1-x^2}]^2}\right)-\dfrac{1}{2}\times\dfrac{1}{1-x^2}\times(0-2x)$

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{x}{1-x^2}+\cos^{-1}x\left(\dfrac{1-x^2+x^2}{(1-x^2)^{3/2}}\right)+\dfrac{x}{1-x^2}$ $\quad\therefore \dfrac{dy}{dx}=\dfrac{\cos^{-1}x}{(1-x^2)^{3/2}}.$

Q17. We've $y=\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}+\log\sqrt{1-x^2}$ $\quad\Rightarrow y=\left(\dfrac{x}{\sqrt{1-x^2}}\right)\sin^{-1}x+\dfrac{1}{2}\log(1-x^2)$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x}{\sqrt{1-x^2}}\right)\times\frac{1}{\sqrt{1-x^2}}+\sin^{-1}x\left(\frac{\sqrt{1-x^2}\times 1-x\times\frac{-2x}{2\sqrt{1-x^2}}}{[\sqrt{1-x^2}]^2}\right)+\frac{1}{2}\times\frac{-2x}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{1-x^2}+\sin^{-1}x\left(\frac{1-x^2+x^2}{[1-x^2]^{3/2}}\right)-\frac{x}{1-x^2} \qquad \therefore \frac{dy}{dx}=\frac{\sin^{-1}x}{[1-x^2]^{3/2}}$$

Q18. (a) Given $\log\sqrt{x^2+y^2}=\tan^{-1}\left(\frac{y}{x}\right)$ $\quad\Rightarrow \frac{1}{2}\log(x^2+y^2)=\tan^{-1}\left(\frac{y}{x}\right)$

$$\Rightarrow \frac{1}{2}\times\frac{(2x+2yy')}{x^2+y^2}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\left(\frac{xy'-y.1}{x^2}\right) \qquad \Rightarrow \frac{x+yy'}{x^2+y^2}=\frac{xy'-y}{x^2+y^2}$$

$$\Rightarrow \frac{x+y}{x^2+y^2}=(x-y)\frac{y'}{x^2+y^2} \qquad \therefore \frac{dy}{dx}=\frac{x+y}{x-y}$$

(b) Given $\log\sqrt{x^2+y^2}=\tan^{-1}\left(\frac{x}{y}\right)$ $\quad\Rightarrow \frac{1}{2}\log(x^2+y^2)=\tan^{-1}\left(\frac{x}{y}\right)$

$$\Rightarrow \frac{1}{2}\times\frac{2x+2y\frac{dy}{dx}}{x^2+y^2}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\times\frac{d}{dx}\left(\frac{x}{y}\right) \qquad \Rightarrow \frac{x+y\frac{dy}{dx}}{x^2+y^2}=\frac{y^2}{y^2+x^2}\times\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow \frac{x+y\frac{dy}{dx}}{x^2+y^2}=\frac{y-x\frac{dy}{dx}}{y^2+x^2} \qquad \Rightarrow x+y\frac{dy}{dx}=y-x\frac{dy}{dx}$$

$$\Rightarrow y\frac{dy}{dx}+x\frac{dy}{dx}=y-x \qquad \Rightarrow (y+x)\frac{dy}{dx}=y-x \qquad \therefore \frac{dy}{dx}=\frac{y-x}{y+x}.$$

Q19. We have $xy\log(x+y)=1$ $\quad\Rightarrow \log(x+y)=\frac{1}{xy}$

$$\Rightarrow \frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=-\frac{1}{x^2y^2}\left(x\frac{dy}{dx}+y.1\right) \qquad \Rightarrow \frac{1}{x+y}+\frac{y'}{x+y}=-\frac{y'}{xy^2}-\frac{1}{x^2y}$$

$$\Rightarrow \frac{dy}{dx}\times\left(\frac{1}{x+y}+\frac{1}{xy^2}\right)=-\left(\frac{1}{x^2y}+\frac{1}{x+y}\right) \qquad \Rightarrow \frac{dy}{dx}\times\left(\frac{xy^2+x+y}{xy^2(x+y)}\right)=-\left(\frac{x+y+x^2y}{x^2y(x+y)}\right)$$

$$\Rightarrow \frac{dy}{dx}\times\left(\frac{xy^2+x+y}{y}\right)=-\left(\frac{x+y+x^2y}{x}\right) \qquad \therefore \frac{dy}{dx}=-\frac{y}{x}\left(\frac{x^2y+x+y}{xy^2+x+y}\right)$$

Q20. We have $y=(x-1)\log(x-1)-(x+1)\log(x+1)$

$$\Rightarrow \frac{dy}{dx}=(x-1)\times\frac{1}{(x-1)}+\log(x-1)\times(1-0)-(x+1)\times\frac{1}{(x+1)}-\log(x+1)\times(1+0)$$

$$\Rightarrow \frac{dy}{dx}=1+\log(x-1)-1-\log(x+1) \qquad \Rightarrow \frac{dy}{dx}=\log(x-1)-\log(x+1)$$

Hence $\frac{dy}{dx}=\log\left(\frac{x-1}{x+1}\right)$.

Q21. Put $x=\tan\theta\Rightarrow\theta=\tan^{-1}x\ \ldots(i)$

$\therefore y=\tan^{-1}[\sqrt{1+x^2}-x]$ $\quad\Rightarrow y=\tan^{-1}[\sqrt{1+\tan^2\theta}-\tan\theta]$

$\Rightarrow y=\tan^{-1}[\sec\theta-\tan\theta]$ $\quad\Rightarrow y=\tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right)=\tan^{-1}\left(\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right)=\tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)$ $\quad\Rightarrow y=\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$\Rightarrow y=\frac{\pi}{4}-\frac{\theta}{2}$ $\quad$ By (i), $y=\frac{\pi}{4}-\frac{1}{2}\tan^{-1}x$ $\quad\therefore \frac{dy}{dx}=-\frac{1}{2(x^2+1)}$.

Q22. We have $y=(\sin^{-1}x)^2+(\cos^{-1}x)^2$ $\quad\Rightarrow y=(\sin^{-1}x+\cos^{-1}x)^2-2(\sin^{-1}x\cos^{-1}x)$

$\Rightarrow y=\left(\frac{\pi}{2}\right)^2-2(\sin^{-1}x\cos^{-1}x)$ $\quad\Rightarrow y_1=0-2\sin^{-1}x\times\frac{-1}{\sqrt{1-x^2}}-2\cos^{-1}x\times\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\,y_1=2\sin^{-1}x-2\cos^{-1}x$ $\quad\Rightarrow \sqrt{1-x^2}\,y_2-\frac{2xy_1}{2\sqrt{1-x^2}}=\frac{2}{\sqrt{1-x^2}}+\frac{2}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)y_2-xy_1=2+2$ $\quad\therefore (1-x^2)y_2-xy_1-4=0$

Q23. Put $x=\cos\theta\Rightarrow\theta=\cos^{-1}x\ldots$(i)

$\therefore\ y=\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)+\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$

$\Rightarrow y=\tan^{-1}\left(\frac{\cos\theta}{1+\sqrt{1-\cos^2\theta}}\right)+\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)\right]$

$\Rightarrow y=\tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right)+\sin\left[2\tan^{-1}\left(\frac{\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}}\right)\right]$

$\Rightarrow y=\tan^{-1}\left(\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}\right)+\sin\left[2\tan^{-1}\tan\frac{\theta}{2}\right]$

$\Rightarrow y=\tan^{-1}\left(\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right)+\sin\left[2\times\frac{\theta}{2}\right]$ $\quad\Rightarrow y=\tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)+\sin\theta$

$\Rightarrow y=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]+\sqrt{1-\cos^2\theta}$ $\quad\Rightarrow y=\frac{\pi}{4}-\frac{\theta}{2}+\sqrt{1-x^2}$

By (i), $y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x+\sqrt{1-x^2}$ $\quad\Rightarrow \frac{dy}{dx}=0-\frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right)+\frac{1}{2\sqrt{1-x^2}}\times(0-2x)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}$ $\quad\therefore \frac{dy}{dx}=\frac{1-2x}{2\sqrt{1-x^2}}$

Q24. We have $y=\sin(p\sin^{-1}x)$ $\quad\Rightarrow y_1=\cos(p\sin^{-1}x)\times p\times\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y_1 = p\cos(p\sin^{-1}x)$ $\quad\Rightarrow \sqrt{1-x^2}\, y_2 + y_1 \times \dfrac{-2x}{2\sqrt{1-x^2}} = -p^2 \sin(p\sin^{-1}x) \times \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\,y_2 - xy_1 = -p^2 y$ $\quad\therefore (1-x^2)\,y_2 - xy_1 + p^2 y = 0$

Q25. We have $x = \sin\left(\dfrac{\sin^{-1}y}{m}\right)$ $\quad\Rightarrow \sin^{-1}x = \sin^{-1}\sin\left(\dfrac{\sin^{-1}y}{m}\right)$

$\Rightarrow \sin^{-1}x = \dfrac{\sin^{-1}y}{m}$ $\quad\Rightarrow m\sin^{-1}x = \sin^{-1}y$

$\Rightarrow \sin(m\sin^{-1}x) = \sin\sin^{-1}y$ $\quad\Rightarrow y = \sin(m\sin^{-1}x)$

You can note that this question is now similar to the previous question.
Therefore, proceed same as in the last sum.

Q26. **(a)** Given $y = e^{a\cos^{-1}x}$ $\Rightarrow y' = e^{a\cos^{-1}x} \times a\left(-\dfrac{1}{\sqrt{1-x^2}}\right)$ $\quad\Rightarrow y'\sqrt{1-x^2} = -ae^{a\cos^{-1}x}$

$\Rightarrow y''\sqrt{1-x^2} - \dfrac{xy'}{\sqrt{1-x^2}} = -ae^{a\cos^{-1}x} \times a\left(-\dfrac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow y''(1-x^2) - xy' = a^2 e^{a\cos^{-1}x}$ $\quad\therefore (1-x^2)y'' - xy' - a^2 y = 0.$

(b) Given $y = e^{m\sin^{-1}x}$ …(i) $\quad\Rightarrow \dfrac{dy}{dx} = e^{m\sin^{-1}x} \times \dfrac{m}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\,\dfrac{dy}{dx} = me^{m\sin^{-1}x}$ $\quad\Rightarrow \sqrt{1-x^2}\,\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} \times \left(\dfrac{-x}{\sqrt{1-x^2}}\right) = m^2 e^{m\sin^{-1}x} \times \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - m^2 y = 0$ $\quad$ [by (i).

Q27. We have $y = e^x \tan^{-1}x$ $\quad\Rightarrow y_1 = e^x \times \dfrac{1}{1+x^2} + \tan^{-1}x \times e^x$

$\Rightarrow (1+x^2)y_1 = e^x + (1+x^2)e^x \tan^{-1}x$ $\quad\Rightarrow (1+x^2)y_1 = e^x + (1+x^2)y$ …(i)

$\Rightarrow (1+x^2)y_2 + 2xy_1 = e^x + (1+x^2)y_1 + 2xy$

$\Rightarrow (1+x^2)y_2 + 2xy_1 = (1+x^2)y_1 - (1+x^2)y + (1+x^2)y_1 + 2xy$

$\Rightarrow (1+x^2)y_2 + 2xy_1 = (2+2x^2)y_1 - (1+x^2-2x)y$ $\quad\therefore (1+x^2)y_2 - 2(1-x+x^2)y_1 + (1-x)^2 y = 0.$

Q28. We have $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$ $\quad\Rightarrow y = (\cos x)^y$ $\quad\Rightarrow \log y = y\log(\cos x)$

$\Rightarrow \dfrac{1}{y} \times \dfrac{dy}{dx} = y\left(\dfrac{-\sin x}{\cos x}\right) + \log(\cos x) \times \dfrac{dy}{dx}$ $\quad\Rightarrow \left(\dfrac{1}{y} - \log\cos x\right) \times \dfrac{dy}{dx} = -y\tan x$

$\Rightarrow (1 - y\log\cos x) \times \dfrac{dy}{dx} = -y^2 \tan x$ $\quad\therefore \dfrac{dy}{dx} = -\dfrac{y^2 \tan x}{1 - y\log\cos x}$

Q29. Put $\tan^{-1}2x = \theta \Rightarrow \tan\theta = 2x \Rightarrow \sin\theta = \dfrac{2x}{\sqrt{1+4x^2}}$ …(i)

Now we have $y = \sin(\tan^{-1}2x)$ $\quad\Rightarrow y = \sin\theta = \dfrac{2x}{\sqrt{1+4x^2}}$ $\quad$ [By (i)

$\therefore \dfrac{dy}{dx} = \dfrac{\sqrt{1+4x^2} \times 2 - 2x \times \dfrac{8x}{2\sqrt{1+4x^2}}}{[\sqrt{1+4x^2}\,]^2}$ $\quad\Rightarrow \dfrac{dy}{dx} = \dfrac{2(1+4x^2) - 8x^2}{[1+4x^2]\sqrt{1+4x^2}}$

$\Rightarrow \frac{dy}{dx}=\frac{2+8x^2-8x^2}{[1+4x^2]^{3/2}}$ $\therefore \frac{dy}{dx}=\frac{2}{[1+4x^2]^{3/2}}$

Q30. (a) We have $y=\sin^{-1}x$ $\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$ $\Rightarrow \sqrt{1-x^2}\times\frac{dy}{dx}=1$

$\Rightarrow \sqrt{1-x^2}\times\frac{d^2y}{dx^2}+\frac{dy}{dx}\times\frac{-2x}{2\sqrt{1-x^2}}=0$ $\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$ $\therefore (1-x^2)y_2-xy_1=0$.

(b) We have $y=(\sin^{-1}x)^2$ $\Rightarrow \frac{dy}{dx}=2(\sin^{-1}x)\times\frac{1}{\sqrt{1-x^2}}$ $\Rightarrow \sqrt{1-x^2}\times\frac{dy}{dx}=2(\sin^{-1}x)$

$\Rightarrow \sqrt{1-x^2}\times\frac{d^2y}{dx^2}+\frac{dy}{dx}\times\frac{-2x}{2\sqrt{1-x^2}}=2\times\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$ $\therefore (1-x^2)y_2=xy_1+2$.

(c) Do yourself. Proceed as in Q30 (b).

(d) Here $y=(3\cot^{-1}x)^2$ $\Rightarrow y=9(\cot^{-1}x)^2$ $\Rightarrow \frac{dy}{dx}=18\cot^{-1}x\times\frac{-1}{1+x^2}$

$\Rightarrow (1+x^2)\frac{dy}{dx}=-18\cot^{-1}x$ $\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=-18\times\frac{-1}{1+x^2}$

$\therefore (x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}=18$.

Q31. Given $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}$ $\Rightarrow \frac{dy}{dx}=0+\frac{1}{1!}+\frac{2x}{2!}+\frac{3x^2}{3!}+...+\frac{nx^{n-1}}{n!}$

$\Rightarrow \frac{dy}{dx}=\left(1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}\right)-\frac{x^n}{n!}$ $\Rightarrow \frac{dy}{dx}=y-\frac{x^n}{n!}$ $\therefore \frac{dy}{dx}-y+\frac{x^n}{n!}=0$

Q32. Given $y=\sqrt{1+\sqrt{1+x^4}}$ $\Rightarrow y^2=1+\sqrt{1+x^4}$ ...(i)

$\Rightarrow 2yy'=0+\frac{1}{2\sqrt{1+x^4}}\times(0+4x^3)$ $\Rightarrow yy'=\frac{x^3}{\sqrt{1+x^4}}$

$\Rightarrow \frac{dy}{dx}=\frac{x^3}{y\sqrt{1+x^4}}$ $\therefore \frac{dy}{dx}=\frac{x^3}{y[y^2-1]}$ [By (i), $y^2-1=\sqrt{1+x^4}$

Q33. We've $\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}=6$ $\Rightarrow \left(\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}\right)^2=6^2$ $\Rightarrow \frac{y}{x}+\frac{x}{y}+2=36$

$\Rightarrow y^2+x^2=34xy$ $\Rightarrow 2yy_1+2x=34[xy_1+y.1]$ $\Rightarrow yy_1+x=17[xy_1+y]$

$\Rightarrow y_1(y-17x)=17y-x$ $\Rightarrow y_1=\frac{17y-x}{y-17x}$ $\therefore \frac{dy}{dx}=\frac{x-17y}{17x-y}$

Q34. We have $x\sqrt{1+y}+y\sqrt{1+x}=0$ $\Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$

$\Rightarrow x^2(1+y)=y^2(1+x)$ $\Rightarrow x^2+yx^2=y^2+xy^2$

$\Rightarrow x^2-y^2=xy^2-yx^2$ $\Rightarrow (x-y)(x+y)=xy(y-x)$

$\Rightarrow (x-y)(x+y)+xy(x-y)=0$ $\Rightarrow (x-y)(x+y+xy)=0$

$\therefore x-y=0$ or $x+y+xy=0$

$\Rightarrow y=x$ is rejected, $\therefore y=-\frac{x}{1+x}$ $\Rightarrow y=-\frac{1+x-1}{1+x}$ $\Rightarrow y=-1+\frac{1}{1+x}$

$\Rightarrow \frac{dy}{dx} = 0 - \frac{1}{(1+x)^2} \times (0+1)$ $\quad \therefore \frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Q35. Given $y = \sqrt{\frac{1-x}{1+x}}$ $\quad \Rightarrow \log y = \log\sqrt{\frac{1-x}{1+x}}$

$\Rightarrow \log y = \frac{1}{2}[\log(1-x) - \log(1+x)]$ $\quad \Rightarrow \frac{1}{y} \times \frac{dy}{dx} = \frac{1}{2}\left[-\frac{1}{1-x} - \frac{1}{1+x}\right]$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$ $\quad \Rightarrow \frac{dy}{dx} = -\frac{y}{2}\left[\frac{1+x+1-x}{1^2 - x^2}\right]$

$\Rightarrow \frac{dy}{dx} = -y\left[\frac{1}{1-x^2}\right]$ $\quad \Rightarrow (1-x^2)\frac{dy}{dx} = -y$ $\quad \therefore (1-x^2)\frac{dy}{dx} + y = 0.$

Q36. Let $y = a^x$ $\quad \Rightarrow \log y = \log a^x$ $\quad \Rightarrow \log y = x \log a$

$\Rightarrow \frac{1}{y} \times \frac{dy}{dx} = \log a \times 1$ $\quad \Rightarrow \frac{dy}{dx} = y \log a$ $\quad \therefore \frac{dy}{dx} = a^x \log_e a$

Q37. Here $x = a\sin pt,\ y = b\cos pt$

Differentiating w.r.t. t,

$\frac{dx}{dt} = ap\cos pt,\ \frac{dy}{dt} = -bp\sin pt$ $\quad \Rightarrow \frac{dy}{dx} = -\frac{b}{a}\tan pt$

On differentiating w.r.t. x, we get : $\frac{d^2y}{dx^2} = -\frac{bp\sec^2 pt}{a} \times \frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{bp\sec^2 pt}{a} \times \frac{1}{ap\cos pt} = -\frac{b}{a^2} \times \frac{1}{\cos^2 pt} \times \frac{1}{\cos pt}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a^2} \times \frac{1}{1-\sin^2 pt} \times \frac{b}{y}$ $\quad \Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a^2} \times \frac{1}{1-\frac{x^2}{a^2}} \times \frac{b}{y}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a^2 - x^2} \times \frac{b}{y}$ $\quad \Rightarrow (a^2 - x^2)y\frac{d^2y}{dx^2} = -b^2$ $\quad \therefore (a^2 - x^2)y\frac{d^2y}{dx^2} + b^2 = 0.$

Q38. We have $y = \frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$ $\quad \Rightarrow y = \frac{e^{4x}+1}{e^{4x}-1}$ $\quad \Rightarrow y = \frac{e^{4x}-1+2}{e^{4x}-1} = 1 + \frac{2}{e^{4x}-1}$

$\therefore \frac{dy}{dx} = 0 + \left(-\frac{2e^{4x} \times 4}{(e^{4x}-1)^2}\right)$ $\quad \Rightarrow \frac{dy}{dx} = -\frac{8e^{4x}}{(e^{4x}-1)^2}$ ...(i)

$\Rightarrow \frac{d^2y}{dx^2} = -8\frac{(e^{4x}-1)^2 4e^{4x} - e^{4x} \times 2(e^{4x}-1) \times 4e^{4x}}{(e^{4x}-1)^4}$

$\Rightarrow \frac{d^2y}{dx^2} = -8\frac{4e^{4x}(e^{4x}-1)[e^{4x} - 1 - e^{4x} \times 2]}{(e^{4x}-1)^4}$ $\quad \Rightarrow \frac{d^2y}{dx^2} = \frac{32e^{4x}(e^{4x}+1)}{(e^{4x}-1)^3}$

Now $\frac{d^2y}{dx^2} + 4y\frac{dy}{dx} = \frac{32e^{4x}(e^{4x}+1)}{(e^{4x}-1)^3} + 4\left(1 + \frac{2}{e^{4x}-1}\right)\left(-\frac{8e^{4x}}{(e^{4x}-1)^2}\right)$ $\quad$ [By (i)

$\Rightarrow \quad = \frac{32e^{4x}(e^{4x}+1)}{(e^{4x}-1)^3} - \left(4 + \frac{8}{e^{4x}-1}\right)\left(\frac{8e^{4x}}{(e^{4x}-1)^2}\right)$

$\Rightarrow \quad = \frac{32e^{4x}(e^{4x}+1)}{(e^{4x}-1)^3} - \left(\frac{4e^{4x}+4}{e^{4x}-1}\right)\left(\frac{8e^{4x}}{(e^{4x}-1)^2}\right)$

$$\Rightarrow \quad = \frac{32e^{4x}(e^{4x}+1)}{(e^{4x}-1)^3} - 4\left(\frac{e^{4x}+1}{e^{4x}-1}\right)\left(\frac{8e^{4x}}{(e^{4x}-1)^2}\right)$$

$$\Rightarrow \quad = \frac{32e^{4x}(e^{4x}+1)}{(e^{4x}-1)^3} - \frac{32e^{4x}(e^{4x}+1)}{(e^{4x}-1)^3} = 0 \qquad \therefore y'' + 4yy' = 0.$$

Q39. **(a)** We have $y = ae^{x} + be^{-2x}$ $\Rightarrow e^{2x}y = ae^{3x} + b$ $\Rightarrow e^{2x}y_1 + 2e^{2x}y = 3ae^{3x} + 0$

$\Rightarrow e^{2x}(y_1 + 2y) = 3ae^{3x}$ $\Rightarrow e^{-x}(y_1 + 2y) = 3a$

$\Rightarrow e^{-x}(y_2 + 2y_1) + (y_1 + 2y)e^{-x}(-1) = 0$ $\Rightarrow e^{-x}[y_2 + 2y_1 - y_1 - 2y] = 0$ $\therefore y_2 + y_1 - 2y = 0$.

(b) Given that $y = ae^{2x} + be^{-x}$ $\Rightarrow e^{x}y = ae^{3x} + b$ $\Rightarrow e^{x}y_1 + e^{x}y = 3ae^{3x}$

$\Rightarrow e^{-2x}(y_1 + y) = 3a$ $\Rightarrow e^{-2x}(y_2 + y_1) - 2e^{-2x}(y_1 + y) = 0$

$\Rightarrow y_2 + y_1 - 2y_1 - 2y = 0$ $\therefore \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$.

Q40. Given $y = 3e^{2x} + 2e^{3x}$ $\Rightarrow e^{-3x}y = 3e^{-x} + 2$ $\Rightarrow e^{-3x}y_1 + ye^{-3x}(-3) = 3e^{-x}(-1) + 0$

$\Rightarrow e^{-3x}(y_1 - 3y) = -3e^{-x}$ $\Rightarrow e^{-2x}(y_1 - 3y) = -3$ $\Rightarrow e^{-2x}(y_2 - 3y_1) + (y_1 - 3y)e^{-2x}(-2) = 0$

$\Rightarrow e^{-2x}(y_2 - 3y_1 - 2y_1 + 6y) = 0$ $\therefore \dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$

Q41. **(a)** We've $y = e^{ax}\sin bx$ $\Rightarrow y_1 = be^{ax}\cos bx + ae^{ax}\sin bx$

$\Rightarrow y_1 = be^{ax}\cos bx + ay \ldots$(i) $\Rightarrow y_2 = b[-be^{ax}\sin bx + ae^{ax}\cos bx] + ay_1$

$\Rightarrow y_2 = b[-by + ae^{ax}\cos bx] + ay_1$ $\Rightarrow y_2 = -b^2y + abe^{ax}\cos bx + ay_1$

By (i), $y_2 = -b^2y + a(y_1 - ay) + ay_1$ $\Rightarrow y_2 = -(b^2 + a^2)y + 2ay_1$

$\therefore y_2 - 2ay_1 + (a^2 + b^2)y = 0$

(b) Given $y = e^{ax}\cos bx$ $\Rightarrow \dfrac{dy}{dx} = ae^{ax}\cos bx - be^{ax}\sin bx$ $\Rightarrow \dfrac{dy}{dx} = ay - be^{ax}\sin bx$

$\Rightarrow \dfrac{d^2y}{dx^2} = a\dfrac{dy}{dx} - b[ae^{ax}\sin bx + be^{ax}\cos bx]$ $\Rightarrow \dfrac{d^2y}{dx^2} = a\dfrac{dy}{dx} - abe^{ax}\sin bx - b^2y$

$\Rightarrow \dfrac{d^2y}{dx^2} = a\dfrac{dy}{dx} - a\left(ay - \dfrac{dy}{dx}\right) - b^2y$ $\Rightarrow \dfrac{d^2y}{dx^2} = 2a\dfrac{dy}{dx} - a^2y - b^2y$

$\therefore \dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2 + b^2)y = 0$.

Q42. We have $y = e^{-2x}(A\cos x + B\sin x)$ $\Rightarrow e^{2x}y = A\cos x + B\sin x$

$\Rightarrow e^{2x}y_1 + 2e^{2x}y = -A\sin x + B\cos x$ $\Rightarrow e^{2x}(y_1 + 2y) = -A\sin x + B\cos x$

$\Rightarrow e^{2x}(y_2 + 2y_1) + 2e^{2x}(y_1 + 2y) = -A\cos x - B\sin x$

$\Rightarrow (y_2 + 2y_1 + 2y_1 + 4y) = -e^{-2x}(A\cos x + B\sin x)$

$\Rightarrow y_2 + 4y_1 + 4y = -y$ $\therefore \dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$.

Q43. (a) Given $y = \cot x$ $\Rightarrow \dfrac{dy}{dx} = -\operatorname{cosec}^2 x$ $\Rightarrow \dfrac{d^2y}{dx^2} = -2\operatorname{cosec} x(-\operatorname{cosec} x\cot x)$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2\operatorname{cosec}^2 x\cot x$ $\Rightarrow \dfrac{d^2y}{dx^2} = 2\left(-\dfrac{dy}{dx}\right)y$ $\therefore \dfrac{d^2y}{dx^2} + 2y\dfrac{dy}{dx} = 0$.

(b) We have $y = \cot x$ $\Rightarrow \dfrac{dy}{dx} = -\operatorname{cosec}^2 x$ $\Rightarrow \dfrac{d^2y}{dx^2} = 2\operatorname{cosec}^2 x\cot x$

$\Rightarrow \dfrac{d^2y}{dx^2} = 2\times\dfrac{1}{\sin^2 x}\times y$ $\quad\Rightarrow \sin^2 x\dfrac{d^2y}{dx^2} = 2y$ $\quad\therefore \sin^2 x\dfrac{d^2y}{dx^2} - 2y = 0$

Q44. $y = x + \tan x$ $\quad\Rightarrow y_1 = 1 + \sec^2 x$ $\quad\Rightarrow y_2 = 0 + 2\sec x \sec x \tan x$

$\Rightarrow y_2 = 2\sec^2 x \tan x$ $\quad\Rightarrow y_2 = 2\times\dfrac{1}{\cos^2 x}\times(y - x)$ $\quad[\because y = x + \tan x \Rightarrow y - x = \tan x$

$\Rightarrow \cos^2 x y_2 = 2y - 2x$ $\quad\therefore \cos^2 x\dfrac{d^2y}{dx^2} - 2y + 2x = 0$

Q45. We have $\tan\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) = a$ $\quad\Rightarrow \dfrac{x^2 - y^2}{x^2 + y^2} = \tan^{-1} a$ Applying Componendo & Dividendo,

$\dfrac{x^2 - y^2 + x^2 + y^2}{x^2 - y^2 - x^2 - y^2} = \dfrac{\tan^{-1} a + 1}{\tan^{-1} a - 1}$ $\quad\Rightarrow \dfrac{2x^2}{-2y^2} = \dfrac{\tan^{-1} a + 1}{\tan^{-1} a - 1}$ $\quad\Rightarrow \dfrac{x^2}{y^2} = -\dfrac{\tan^{-1} a + 1}{\tan^{-1} a - 1}$

$\Rightarrow \dfrac{y^2 2x - x^2(2yy_1)}{y^4} = 0$ $\quad\Rightarrow 2y - 2xy_1 = 0$ $\quad\therefore \dfrac{dy}{dx} = \dfrac{y}{x}$.

Q46. We've $y = \sec x + \tan x$ $\quad\Rightarrow y = \dfrac{1 + \sin x}{\cos x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\cos x(0 + \cos x) - (1 + \sin x)(-\sin x)}{\cos^2 x}$ $\quad\Rightarrow \dfrac{dy}{dx} = \dfrac{\cos^2 x + \sin^2 x + \sin x}{1 - \sin^2 x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1 + \sin x}{(1 - \sin x)(1 + \sin x)}$ $\quad\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1 - \sin x}$

$\Rightarrow \dfrac{d^2y}{dx^2} = -\dfrac{1}{(1 - \sin x)^2}\times(0 - \cos x)$ $\quad\therefore \dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1 - \sin x)^2}$.

Q47. We've $y = \sqrt{\dfrac{1 + \tan x}{1 - \tan x}}$ $\quad\Rightarrow \log y = \log\sqrt{\dfrac{1 + \tan x}{1 - \tan x}}$

$\Rightarrow \log y = \dfrac{1}{2}\left(\log[1 + \tan x] - \log[1 - \tan x]\right)$ $\quad\Rightarrow \dfrac{1}{y}\times\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{\sec^2 x}{1 + \tan x} - \dfrac{-\sec^2 x}{1 - \tan x}\right)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{2}\sec^2 x\left(\dfrac{1}{1 + \tan x} + \dfrac{1}{1 - \tan x}\right)$ $\quad\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{2}\sec^2 x\left(\dfrac{2}{1 - \tan^2 x}\right)$

$\Rightarrow \dfrac{dy}{dx} = y\left(\dfrac{1 + \tan^2 x}{1 - \tan^2 x}\right)$ $\quad\Rightarrow \dfrac{dy}{dx} = y\left(\dfrac{1}{\dfrac{1 - \tan^2 x}{1 + \tan^2 x}}\right)$ $\quad\Rightarrow \dfrac{dy}{dx} = y\left(\dfrac{1}{\cos 2x}\right)$

$\therefore \dfrac{dy}{dx} = y\sec 2x$.

Q48. Given $y = \sqrt{\dfrac{1 - \sin 2x}{1 + \sin 2x}}$ $\quad\Rightarrow y = \sqrt{\dfrac{\cos^2 x + \sin^2 x - 2\sin x\cos x}{\cos^2 x + \sin^2 x + 2\sin x\cos x}}$

$\Rightarrow y = \sqrt{\dfrac{(\cos x - \sin x)^2}{(\cos x + \sin x)^2}}$ $\quad\Rightarrow y = \dfrac{\cos x - \sin x}{\cos x + \sin x} = \dfrac{1 - \tan x}{1 + \tan x}$ $\quad\Rightarrow y = \tan\left(\dfrac{\pi}{4} - x\right)$

$\Rightarrow \dfrac{dy}{dx} = \sec^2\left(\dfrac{\pi}{4} - x\right)\times(0 - 1)$ $\quad\therefore \dfrac{dy}{dx} + \sec^2\left(\dfrac{\pi}{4} - x\right) = 0$.

Q49. We've $y = A\cos\log x + B\sin\log x$ $\quad\Rightarrow y_1 = -A\sin\log x\times\dfrac{1}{x} + B\cos\log x\times\dfrac{1}{x}$

$\Rightarrow xy_1 = -A\sin\log x + B\cos\log x$ $\Rightarrow xy_2 + y_1 \times 1 = -A\cos\log x \times \frac{1}{x} - B\sin\log x \times \frac{1}{x}$

$\Rightarrow x^2y_2 + xy_1 = -(A\cos\log x + B\sin\log x)$ $\Rightarrow x^2y_2 + xy_1 = -y$ $\therefore x^2y_2 + xy_1 + y = 0$.

Q50. We have $y = \cos\cos x$ $\Rightarrow y_1 = -\sin\cos x(-\sin x)$ $\Rightarrow y_1 = \sin x \sin\cos x$

$\Rightarrow y_2 = \sin x \times [\cos\cos x \times (-\sin x)] + \sin\cos x \times (\cos x)$

$\Rightarrow y_2 = \sin x \times [-y\sin x] + \sin\cos x \times (\cos x)$

$\Rightarrow y_2 = -y\sin^2 x + \frac{y_1}{\sin x} \times (\cos x)$ $[\because y_1 = \sin x\sin\cos x \Rightarrow \frac{y_1}{\sin x} = \sin\cos x$

$\therefore y_2 - y_1\cot x + y\sin^2 x = 0$

Q51. We have $y = x\cos y$ $\Rightarrow y_1 = x[-\sin y \times y_1] + \cos y \times 1$ $\Rightarrow y_1 = -xy_1\sin y + \cos y$

$\Rightarrow y_1[1 + x\sin y] = \cos y$ $\Rightarrow y_1 = \frac{\cos y}{1 + x\sin y} \times \frac{x}{x}$ $\therefore \frac{dy}{dx} = \frac{y}{x[1 + x\sin y]}$

Q52. Given $\cos y = x\cos(a+y)$ $\Rightarrow x = \frac{\cos y}{\cos(a+y)}$

Diff. w.r.t. y both sides, we get $\frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) - \cos y(-\sin(a+y))}{\cos^2(a+y)}$

$\Rightarrow \frac{dx}{dy} = \frac{\cos y\sin(a+y) - \cos(a+y)\sin y}{\cos^2(a+y)}$ $\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)}$ $\therefore \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Now $\sin a\frac{dy}{dx} = \cos^2(a+y)$ $\Rightarrow \sin a\frac{d^2y}{dx^2} = -2\cos(a+y)\sin(a+y)\frac{dy}{dx}$

$\Rightarrow \sin a\frac{d^2y}{dx^2} = -\sin 2(a+y)\frac{dy}{dx}$ $\therefore \sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$.

Q53. We've $\sin y = x\sin(a+y)$

Differentiating w.r.t. x both sides, we get : $y_1\cos y = x\cos(a+y)y_1 + \sin(a+y) \times 1$

$\Rightarrow y_1[\cos y - x\cos(a+y)] = \sin(a+y)$ $\Rightarrow y_1\left[\cos y - \frac{\sin y}{\sin(a+y)}\cos(a+y)\right] = \sin(a+y)$

$\Rightarrow y_1\left[\frac{\cos y\sin(a+y) - \sin y\cos(a+y)}{\sin(a+y)}\right] = \sin(a+y)$

$\Rightarrow y_1\left[\frac{\sin(a+y-y)}{\sin(a+y)}\right] = \sin(a+y)$ $\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Q54. We've $y = \left[x + \sqrt{x^2+a^2}\right]^n$ $\Rightarrow \frac{dy}{dx} = n\left[x + \sqrt{x^2+a^2}\right]^{n-1} \times \left(1 + \frac{2x}{2\sqrt{x^2+a^2}}\right)$

$\Rightarrow \frac{dy}{dx} = n\frac{\left[x + \sqrt{x^2+a^2}\right]^n}{\left[x + \sqrt{x^2+a^2}\right]} \times \left(\frac{x + \sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$ $\Rightarrow \frac{dy}{dx} = n\frac{y}{\sqrt{x^2+a^2}}$ $\therefore \frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$

Q55. We've $y = \left[x + \sqrt{x^2+a^2}\right]^m$ $\Rightarrow \frac{dy}{dx} = m\left[x + \sqrt{x^2+a^2}\right]^{m-1} \times \left(1 + \frac{2x}{2\sqrt{x^2+a^2}}\right)$

$\Rightarrow \frac{dy}{dx} = m\frac{\left[x + \sqrt{x^2+a^2}\right]^m}{\left[x + \sqrt{x^2+a^2}\right]} \times \left(\frac{x + \sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$ $\Rightarrow \frac{dy}{dx} = \frac{my}{\sqrt{x^2+a^2}}$ $\Rightarrow \sqrt{x^2+a^2}\frac{dy}{dx} = my$

$\Rightarrow \sqrt{x^2+a^2}\frac{d^2y}{dx^2}+\frac{dy}{dx}\times\frac{2x}{2\sqrt{x^2+a^2}}=m\times\frac{dy}{dx}$ $\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=m\sqrt{x^2+a^2}\frac{dy}{dx}$

$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=m\times my$ $\therefore (x^2+a^2)y_2+xy_1-m^2y=0$

Q56. Proceed same as in Q55.

Q57. We have $y=\sin^{-1}x$ $\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$ $\Rightarrow \sqrt{1-x^2}\frac{dy}{dx}=1$

$\Rightarrow \sqrt{1-x^2}\frac{d^2y}{dx^2}+\frac{dy}{dx}\times\frac{-2x}{2\sqrt{1-x^2}}=0$ $\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{x\frac{dy}{dx}}{1-x^2}$ $\Rightarrow \frac{d^2y}{dx^2}=\frac{x\times\frac{1}{\sqrt{1-x^2}}}{1-x^2}$ $[\because \sin^{-1}x=y\Rightarrow \sin y=x$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{\sin y}{[1-\sin^2 y]^{3/2}}$ $\Rightarrow \frac{d^2y}{dx^2}=\frac{\sin y}{[\cos^2 y]^{3/2}}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{\sin y}{\cos^3 y}$ $\Rightarrow \frac{d^2y}{dx^2}=\frac{\sin y}{\cos y}\times\frac{1}{\cos^2 y}$ $\therefore \frac{d^2y}{dx^2}=\sec^2 y\tan y$

Q58. Given $y=\log\sqrt{\frac{1-\cos x}{1+\cos x}}$ $\Rightarrow y=\frac{1}{2}\left[\log(1-\cos x)-\log(1+\cos x)\right]$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[\frac{-(-\sin x)}{1-\cos x}-\frac{-\sin x}{1+\cos x}\right]$ $\Rightarrow \frac{dy}{dx}=\frac{\sin x}{2}\left[\frac{1}{1-\cos x}+\frac{1}{1+\cos x}\right]$

$\Rightarrow \frac{dy}{dx}=\frac{\sin x}{2}\left[\frac{2}{1-\cos^2 x}\right]$ $\Rightarrow \frac{dy}{dx}=\cos ecx$

$\Rightarrow \frac{d^2y}{dx^2}=-\cos ecx\cot x$ $\Rightarrow -\frac{d^3y}{dx^3}=\cos ecx(-\cos ec^2x)+\cot x(-\cos ecx\cot x)$

$\Rightarrow -\frac{d^3y}{dx^3}=-\cos ec^3x-\cot^2 x\cos ecx$ $\therefore \frac{d^3y}{dx^3}=\operatorname{cosec}x[\cot^2 x+\operatorname{cosec}^2 x]$

Q59. Put $x=\sin\alpha, y=\sin\beta\Rightarrow \alpha=\sin^{-1}x, \beta=\sin^{-1}y$ …(i)

$\therefore \sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ $\Rightarrow \sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta)$

$\Rightarrow \cos\alpha+\cos\beta=2a\left(\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right)$ $\Rightarrow 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=2a\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\Rightarrow \cot\frac{\alpha-\beta}{2}=a$ $\Rightarrow \alpha-\beta=2\cot^{-1}a$ $\Rightarrow \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$ [By (i)

$\therefore \frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\times\frac{dy}{dx}=0$ i.e., $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Q60. We've $\sqrt{1+x^2}+\sqrt{1+y^2}=a(x-y)$ $\Rightarrow \frac{2x}{2\sqrt{1+x^2}}+\frac{2yy'}{2\sqrt{1+y^2}}=a(1-y')$

$\Rightarrow \frac{x}{\sqrt{1+x^2}}+\frac{yy'}{\sqrt{1+y^2}}=a(1-y')$ Using $a=\frac{\sqrt{1+x^2}+\sqrt{1+y^2}}{(x-y)}$

$\Rightarrow \frac{x}{\sqrt{1+x^2}}+\frac{yy'}{\sqrt{1+y^2}}=\frac{\sqrt{1+x^2}+\sqrt{1+y^2}}{(x-y)}(1-y')$

$$\Rightarrow \frac{x}{\sqrt{1+x^2}}+\frac{yy'}{\sqrt{1+y^2}}=\frac{\sqrt{1+x^2}+\sqrt{1+y^2}}{(x-y)}-\frac{\sqrt{1+x^2}+\sqrt{1+y^2}}{(x-y)}y'$$

$$\Rightarrow y'\left[\frac{y}{\sqrt{1+y^2}}+\frac{\sqrt{1+x^2}+\sqrt{1+y^2}}{(x-y)}\right]=\frac{\sqrt{1+x^2}+\sqrt{1+y^2}}{(x-y)}-\frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow y'\left[\frac{xy-y^2+\sqrt{(1+y^2)(1+x^2)}+1+y^2}{(x-y)\sqrt{1+y^2}}\right]=\frac{1+x^2+\sqrt{(1+y^2)(1+x^2)}-x^2+xy}{(x-y)\sqrt{1+x^2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1+\sqrt{(1+y^2)(1+x^2)}+xy}{xy+\sqrt{(1+y^2)(1+x^2)}+1}\times\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}} \qquad \therefore \frac{dy}{dx}=\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}$$

Alternative Method : Put $x=i\sin\alpha,\ y=i\sin\beta \Rightarrow \alpha=\sin^{-1}\left(\frac{x}{i}\right),\beta=\sin^{-1}\left(\frac{y}{i}\right)\ldots(i)$

$$\therefore \sqrt{1+x^2}+\sqrt{1+y^2}=a(x-y) \qquad \Rightarrow \sqrt{1+i^2\sin^2\alpha}+\sqrt{1+i^2\sin^2\beta}=a(\sin\alpha-\sin\beta)$$

$$\Rightarrow \sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta) \qquad \Rightarrow \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$$

$$\Rightarrow 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=a\left(2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right) \qquad \Rightarrow \cot\frac{\alpha-\beta}{2}=a \qquad \Rightarrow \alpha-\beta=2\cot^{-1}a$$

By (i), $\sin^{-1}\left(\frac{x}{i}\right)-\sin^{-1}\left(\frac{y}{i}\right)=2\cot^{-1}a \qquad \Rightarrow \frac{1}{\sqrt{1-\frac{x^2}{i^2}}}\times\left(\frac{1}{i}\right)-\frac{1}{\sqrt{1-\frac{y^2}{i^2}}}\times\left(\frac{1}{i}\right)\times\frac{dy}{dx}=0$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}}-\frac{1}{\sqrt{1+y^2}}\frac{dy}{dx}=0 \qquad \therefore \frac{dy}{dx}=\sqrt{\frac{1+y^2}{1+x^2}}$$

Q61. Put $x=\sin\alpha,\ y=\sin\beta \Rightarrow \alpha=\sin^{-1}x,\beta=\sin^{-1}y\ldots(i)$

$$\therefore x\sqrt{1-y^2}-y\sqrt{1-x^2}=a \qquad \Rightarrow \sin\alpha\sqrt{1-\sin^2\beta}-\sin\beta\sqrt{1-\sin^2\alpha}=a$$

$$\Rightarrow \sin\alpha\cos\beta-\sin\beta\cos\alpha=a \qquad \Rightarrow \sin(\alpha-\beta)=a \qquad \Rightarrow \alpha-\beta=\sin^{-1}a$$

By (i), $\sin^{-1}x-\sin^{-1}y=\sin^{-1}a \qquad \Rightarrow \frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\times\frac{dy}{dx}=0$

$$\Rightarrow -\frac{1}{\sqrt{1-y^2}}\times\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}} \qquad \therefore \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

Q62. Put $x^2=\sin\alpha,\ y^2=\sin\beta \Rightarrow \alpha=\sin^{-1}x^2,\beta=\sin^{-1}y^2\ldots(i)$

$$\therefore \sqrt{1-x^4}+\sqrt{1-y^4}=\lambda(x^2-y^2) \qquad \Rightarrow \sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=\lambda(\sin\alpha-\sin\beta)$$

$$\Rightarrow \cos\alpha+\cos\beta=\lambda(\sin\alpha-\sin\beta) \qquad \Rightarrow 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=2\lambda\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \cot\frac{\alpha-\beta}{2}=\lambda \qquad \Rightarrow \alpha-\beta=2\cot^{-1}\lambda \qquad \Rightarrow \sin^{-1}x^2-\sin^{-1}y^2=2\cot^{-1}\lambda \quad \text{By (i)}$$

$$\Rightarrow \frac{2x}{\sqrt{1-x^4}}-\frac{2y}{\sqrt{1-y^4}}\times\frac{dy}{dx}=0 \qquad \therefore \frac{dy}{dx}=\frac{x}{y}\sqrt{\frac{1-y^4}{1-x^4}}$$

Q63. Put $x^3=\sin\alpha,\ y^3=\sin\beta \Rightarrow \alpha=\sin^{-1}x^3,\beta=\sin^{-1}y^3$ and proceed same as in Q62.

Q64. Put $x^n=\sin\alpha,\ y^n=\sin\beta \Rightarrow \alpha=\sin^{-1}x^n,\beta=\sin^{-1}y^n\ldots(i)$

$$\therefore \sqrt{1-x^{2n}}+\sqrt{1-y^{2n}}=a^n(x^n-y^n) \qquad \Rightarrow \sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a^n(\sin\alpha-\sin\beta)$$

$\Rightarrow \cos\alpha + \cos\beta = a^n(\sin\alpha - \sin\beta)$ $\quad\Rightarrow 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2a^n\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\Rightarrow \cot\frac{\alpha-\beta}{2} = a^n$ $\quad\Rightarrow \alpha-\beta = 2\cot^{-1}a^n$ $\quad\Rightarrow \sin^{-1}x^n - \sin^{-1}y^n = 2\cot^{-1}a^n$ By (i)

$\Rightarrow \frac{nx^{n-1}}{\sqrt{1-x^{2n}}} - \frac{ny^{n-1}}{\sqrt{1-y^{2n}}}\times\frac{dy}{dx} = 0$ $\quad\Rightarrow -\frac{y^{n-1}}{\sqrt{1-y^{2n}}}\times\frac{dy}{dx} = -\frac{x^{n-1}}{\sqrt{1-x^{2n}}}$

$\Rightarrow \frac{dy}{dx} = \frac{x^{n-1}}{y^{n-1}}\times\frac{\sqrt{1-y^{2n}}}{\sqrt{1-x^{2n}}}$ $\quad\therefore \frac{dy}{dx} = \left(\frac{x}{y}\right)^{n-1}\sqrt{\frac{1-y^{2n}}{1-x^{2n}}}$

NOTE : Q64 can be kept in mind as a **general form** of the questions of these forms (like Question No. 59 - Question No. 64*).

Q65. We have $(x-a)^2 + (y-b)^2 = c^2 \ldots$(i) $\quad\Rightarrow 2(x-a)(1-0) + 2(y-b)(y'-0) = 0$

$\Rightarrow (x-a) = -(y-b)y' \ldots$(ii) $\quad\Rightarrow -1 = (y-b)y'' + (y')^2$

$\Rightarrow -\frac{[1+(y')^2]}{y''} = (y-b)$ Replacing value of (y – b) in (ii), $(x-a) = \frac{[1+(y')^2]}{y''}\times y'$

Substituting values of (x – a) and (y – b) in (i), we get :

$\left(\frac{[1+(y')^2]}{y''}\times y'\right)^2 + \left(-\frac{[1+(y')^2]}{y''}\right)^2 = c^2$ $\quad\Rightarrow \left([1+(y')^2]y'\right)^2 + [1+(y')^2]^2 = (cy'')^2$

$\Rightarrow \left[(y')^2+1\right][1+(y')^2]^2 = (cy'')^2$ $\quad\Rightarrow \frac{[1+(y')^2]^3}{y''^2} = c^2$ $\quad\Rightarrow c = \frac{[1+(y')^2]^{3/2}}{y''}$

$\therefore c = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$. Hence $c = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is a constant independent of both a and b.

Q66. Given $\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots$ to $\infty = \frac{\sin x}{x}$

$\Rightarrow \log\cos\frac{x}{2} + \log\cos\frac{x}{2^2} + \log\cos\frac{x}{2^3} + \ldots$ to $\infty = \log\left(\frac{\sin x}{x}\right)$

$\Rightarrow \log\cos\frac{x}{2} + \log\cos\frac{x}{2^2} + \log\cos\frac{x}{2^3} + \ldots$ to $\infty = \log\sin x - \log x$

$\Rightarrow \frac{-\sin\frac{x}{2}}{\cos\frac{x}{2}}\times\frac{1}{2} + \frac{-\sin\frac{x}{2^2}}{\cos\frac{x}{2^2}}\times\frac{1}{2^2} + \frac{-\sin\frac{x}{2^3}}{\cos\frac{x}{2^3}}\times\frac{1}{2^3} + \ldots$ to $\infty = \frac{\cos x}{\sin x} - \frac{1}{x}$

$\Rightarrow -\frac{1}{2}\times\tan\frac{x}{2} - \frac{1}{2^2}\times\tan\frac{x}{2^2} - \frac{1}{2^3}\times\tan\frac{x}{2^3} + \ldots$ to $\infty = \cot x - \frac{1}{x}$

$\Rightarrow -\frac{1}{2}\times\sec^2\frac{x}{2}\times\frac{1}{2} - \frac{1}{2^2}\times\sec^2\frac{x}{2^2}\times\frac{1}{2^2} - \frac{1}{2^3}\times\sec^2\frac{x}{2^3}\times\frac{1}{2^3} + \ldots$ to $\infty = -\text{cosec}^2 x + \frac{1}{x^2}$

$\therefore \frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + \ldots$ to $\infty = \text{cosec}^2 x - \frac{1}{x^2}$.

Q67. We have $y = \sqrt{x^2+1} - \log\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)$ $\quad\Rightarrow y = \sqrt{x^2+1} - \log\left(\frac{1+\sqrt{x^2+1}}{x}\right)$

$\Rightarrow y=\sqrt{x^2+1}-\log[1+\sqrt{x^2+1}]+\log x$ $\Rightarrow \frac{dy}{dx}=\frac{2x}{2\sqrt{x^2+1}}-\frac{1}{1+\sqrt{x^2+1}}\left(0+\frac{2x}{2\sqrt{x^2+1}}\right)+\frac{1}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}\left[1-\frac{1}{1+\sqrt{x^2+1}}\right]+\frac{1}{x}$ $\Rightarrow \frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}\left[\frac{\sqrt{x^2+1}}{1+\sqrt{x^2+1}}\right]+\frac{1}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{x}{1+\sqrt{x^2+1}}+\frac{1}{x}$ $\Rightarrow \frac{dy}{dx}=\frac{x^2+1+\sqrt{x^2+1}}{x[1+\sqrt{x^2+1}]}$

$\Rightarrow \frac{dy}{dx}=\frac{[\sqrt{x^2+1}][\sqrt{x^2+1}]+\sqrt{x^2+1}}{x[1+\sqrt{x^2+1}]}$ $\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x^2+1}\,[\sqrt{x^2+1}]+1]}{x[1+\sqrt{x^2+1}]}$

$\therefore \frac{dy}{dx}=\frac{\sqrt{x^2+1}}{x}$.

Q68. Let $y=f(x)=\left(\frac{3+2x}{1+x}\right)^{2+3x}$ $\Rightarrow \log y=\log\left(\frac{3+2x}{1+x}\right)^{2+3x}$

$\Rightarrow \log y=(2+3x)\left[\log(3+2x)-\log(1+x)\right]$

$\Rightarrow \frac{1}{y}\times\frac{dy}{dx}=(2+3x)\left[\frac{2}{3+2x}-\frac{1}{1+x}\right]+3\log\left(\frac{3+2x}{1+x}\right)$

$\Rightarrow \frac{dy}{dx}=y\left\{(2+3x)\left[\frac{2}{3+2x}-\frac{1}{1+x}\right]+3\log\left(\frac{3+2x}{1+x}\right)\right\}$

$\Rightarrow f'(x)=\left(\frac{3+2x}{1+x}\right)^{2+3x}\left\{(2+3x)\left[\frac{2}{3+2x}-\frac{1}{1+x}\right]+3\log\left(\frac{3+2x}{1+x}\right)\right\}$

$\therefore f'(0)=\left(\frac{3+2\times0}{1+0}\right)^{2+3\times0}\left\{(2+3\times0)\left[\frac{2}{3+2\times0}-\frac{1}{1+0}\right]+3\log\left(\frac{3+2\times0}{1+0}\right)\right\}$

$\Rightarrow f'(0)=9\left\{(2)\left[\frac{2}{3}-1\right]+3\log 3\right\}$ $\Rightarrow f'(0)=9\left\{-\frac{2}{3}+3\log 3\right\}$

$\therefore f'(0)=27\log 3-6$.

Q69. We have $y=f\left(\frac{2x-1}{1+x^2}\right)$

$\Rightarrow \frac{dy}{dx}=f'\left(\frac{2x-1}{1+x^2}\right)\times\frac{(1+x^2)(2-0)-(2x-1)(0+2x)}{(1+x^2)^2}$

$\Rightarrow \frac{dy}{dx}=\sin\left(\frac{2x-1}{1+x^2}\right)^2\times\frac{(1+x^2)2-4x^2+2x}{(1+x^2)^2}$ $[\because f'(x)=\sin x^2$

$\therefore \frac{dy}{dx}=\sin\left(\frac{2x-1}{1+x^2}\right)^2\times\frac{2(1-x^2+x)}{(1+x^2)^2}$

Q70. **(a)** We've $x=\sin\left(\frac{1}{a}\log y\right)$ $\Rightarrow a\sin^{-1}x=\log y$ $\Rightarrow y=e^{a\sin^{-1}x}$

$\Rightarrow \frac{dy}{dx}=e^{a\sin^{-1}x}\times a\times\frac{1}{\sqrt{1-x^2}}$ $\Rightarrow \sqrt{1-x^2}\times\frac{dy}{dx}=ae^{a\sin^{-1}x}$

$\Rightarrow \sqrt{1-x^2}\times\frac{d^2y}{dx^2}+\frac{dy}{dx}\times\frac{-2x}{\sqrt{1-x^2}}=ae^{a\sin^{-1}x}\times a\times\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = a^2y \quad \Rightarrow (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2y = 0$

(b) We have $x = \tan\left(\dfrac{1}{a}\log y\right) \quad \Rightarrow a\tan^{-1}x = \log y \quad \Rightarrow y = e^{a\tan^{-1}x}$ …(i)

$\Rightarrow \dfrac{dy}{dx} = e^{a\tan^{-1}x} \times a \times \dfrac{1}{1+x^2} \quad \Rightarrow (1+x^2)\dfrac{dy}{dx} = ae^{a\tan^{-1}x} \quad \Rightarrow (1+x^2)\dfrac{dy}{dx} = ay$ By (i)

$\Rightarrow (1+x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = a\times\dfrac{dy}{dx} \quad \therefore (1+x^2)\dfrac{d^2y}{dx^2} + [2x-a]\times\dfrac{dy}{dx} = 0$

Q71. Let $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \log\sqrt{\dfrac{x-a}{x+a}} \quad \Rightarrow y = \tan^{-1}\left(\dfrac{x}{a}\right) + \dfrac{1}{2}\left[\log(x-a) - \log(x+a)\right]$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1+\left(\dfrac{x}{a}\right)^2}\times\dfrac{d}{dx}\left(\dfrac{x}{a}\right) + \dfrac{1}{2}\left[\dfrac{1}{x-a} - \dfrac{1}{x+a}\right] \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{a^2}{a^2+x^2}\times\left(\dfrac{1}{a}\right) + \dfrac{1}{2}\left[\dfrac{x+a-x+a}{x^2-a^2}\right]$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{a}{x^2+a^2} + \dfrac{a}{x^2-a^2} \quad \Rightarrow \dfrac{dy}{dx} = a\left(\dfrac{x^2-a^2+x^2+a^2}{x^4-a^4}\right) \quad \therefore \dfrac{dy}{dx} = \dfrac{2ax^2}{x^4-a^4}$.

Q72. **(a)** Given $\dfrac{x}{x-y} = \log\dfrac{a}{x-y} \quad \Rightarrow \dfrac{x}{x-y} = \log a - \log(x-y) \quad \Rightarrow x = (x-y)[\log a - \log(x-y)]$

$\Rightarrow 1 = (x-y)\left[0 - \dfrac{1}{x-y}\left(1-\dfrac{dy}{dx}\right)\right] + [\log a - \log(x-y)]\left(1-\dfrac{dy}{dx}\right)$

$\Rightarrow 1 = -1 + \dfrac{dy}{dx} + \dfrac{x}{x-y}\left(1-\dfrac{dy}{dx}\right) \quad \Rightarrow 2 = \dfrac{dy}{dx}\left(1-\dfrac{x}{x-y}\right) + \dfrac{x}{x-y}$

$\Rightarrow 2 = \dfrac{dy}{dx}\left(\dfrac{x-y-x}{x-y}\right) + \dfrac{x}{x-y} \quad \Rightarrow \dfrac{2(x-y)}{y} = -\dfrac{dy}{dx} + \dfrac{x}{y}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y} - \dfrac{2(x-y)}{y} = \dfrac{1}{y}(x-2x+2y) \quad \therefore \dfrac{dy}{dx} = 2 - \dfrac{x}{y}$.

(b) Method 1 : We've $(x-y).e^{\frac{x}{x-y}} = a$ On differentiating w.r.t. x both the sides, we get :

$(x-y)\times e^{\frac{x}{x-y}}\left(\dfrac{(x-y)\times 1 - x(1-y')}{(x-y)^2}\right) + e^{\frac{x}{x-y}}\times(1-y') = 0$

$\Rightarrow \dfrac{x-y-x+xy'}{x-y} + 1 - y' = 0 \quad \Rightarrow \dfrac{-y+xy'+x-xy'-y+yy'}{x-y} = 0$

$\Rightarrow \dfrac{-2y+x+yy'}{x-y} = 0 \quad \Rightarrow yy' + x = 2y \quad \therefore y\dfrac{dy}{dx} + x = 2y$.

Method 2 : We've $(x-y).e^{\frac{x}{x-y}} = a \quad \Rightarrow \log_e\left\{(x-y).e^{\frac{x}{x-y}}\right\} = \log_e a$

$\Rightarrow \log_e(x-y) + \log_e e^{\frac{x}{x-y}} = \log_e a \quad \Rightarrow \log_e(x-y) + \dfrac{x}{x-y}\log_e e = \log_e a$

$\Rightarrow \dfrac{x}{x-y} = \log_e a - \log_e(x-y)$ (Now proceed as in **Q72-a).**

Q73. Given $e^x + e^y = e^{x+y} \quad \Rightarrow 1 + e^{y-x} = e^y$ ...(i) [Dividing both sides by e^x

Diff. w.r.t. x both sides, $0+\frac{d}{dx}(e^{y-x})=\frac{d}{dx}(e^{y})$ $\Rightarrow e^{y-x}\frac{d}{dx}(y-x)=e^{y}\frac{dy}{dx}$

$\Rightarrow e^{y-x}\left(\frac{dy}{dx}-1\right)=e^{y}\frac{dy}{dx}$ $\Rightarrow \frac{dy}{dx}(e^{y-x}-e^{y})=e^{y-x}$ $\Rightarrow \frac{dy}{dx}(-1)=e^{y-x}$ [Using (i)

$\therefore \frac{dy}{dx}+e^{y-x}=0$.

Q74. We have $y=\sqrt{x}+\frac{1}{\sqrt{x}}$ $\Rightarrow y=\frac{x+1}{\sqrt{x}}$ ...(i) $\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x}(1+0)-(x+1)\times\frac{1}{2\sqrt{x}}}{[\sqrt{x}]^2}$

$\Rightarrow x\frac{dy}{dx}=\sqrt{x}-\frac{x+1}{2\sqrt{x}}$ $\Rightarrow 2x\frac{dy}{dx}=2\sqrt{x}-\frac{x+1}{\sqrt{x}}$ By (i), we get :

$2x\frac{dy}{dx}=2\sqrt{x}-y$ $\Rightarrow 2x\frac{dy}{dx}+y=2\sqrt{x}$ $\therefore 2x\,y'+y=2\sqrt{x}$

Q75. Given $y=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{a^2-b^2}}\times\frac{1}{1+\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)^2}\times\frac{d}{dx}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{a^2-b^2}}\times\frac{1}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}\times\sqrt{\frac{a-b}{a+b}}\times\sec^2\frac{x}{2}\times\frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{a^2-b^2}}\times\frac{a+b}{a+b+(a-b)\tan^2\frac{x}{2}}\times\sqrt{\frac{a-b}{a+b}}\times\sec^2\frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{a+b+(a-b)\tan^2\frac{x}{2}}\times\sec^2\frac{x}{2}$$

$\Rightarrow \frac{dy}{dx}=\frac{\sec^2(x/2)}{a\left(1+\tan^2(x/2)\right)+b\left(1-\tan^2(x/2)\right)}$ Divide Nr & Dr both by $1+\tan^2\frac{x}{2}=\sec^2\frac{x}{2}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{a+b\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right)}$ $\Rightarrow \frac{dy}{dx}=\frac{1}{a+b\cos x}$

$\Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{(a+b\cos x)^2}\times\left(0-b\sin x\right)$ $\therefore \frac{d^2y}{dx^2}=\frac{b\sin x}{(a+b\cos x)^2}$

Q76. We have $x=e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$ $\Rightarrow \log x=\log e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$ $\Rightarrow \log x=\tan^{-1}\left(\frac{y-x^2}{x^2}\right)$

$\Rightarrow \tan\log x=\frac{y-x^2}{x^2}$ $\Rightarrow y=x^2\tan\log x+x^2$ Diff. w.r.t. x both sides,

$D(y)=2x(\tan\log x+1)+x^2(\sec^2\log x\times\frac{1}{x}+0)$

$\Rightarrow D(y)=2x(\tan\log x+1)+x\sec^2\log x$ $\therefore D(y)=x[2(\tan\log x+1)+\sec^2\log x]$

Q77. **(a)** Let $y=(x^x)^x$ $\Rightarrow y=e^{\log(x^x)^x}$ $\Rightarrow y=e^{x\log(x^x)}$ $\Rightarrow y=e^{x^2\log x}$

$\Rightarrow \frac{dy}{dx} = e^{x^2 \log x}\left[x^2 \times \frac{1}{x} + 2x \log x\right]$ $\qquad \therefore \frac{dy}{dx} = (x^x)^x . x[1 + 2\log x]$

(b) Let $y = x^{x^x}$ $\qquad \Rightarrow y = e^{\log x^{x^x}}$ $\qquad \Rightarrow y = e^{x^x \log x}$

$\Rightarrow \frac{dy}{dx} = e^{x^x \log x}\left(x^x \times \frac{1}{x} + x^x(1 + \log x) \times \log x\right)$

$\Rightarrow \frac{dy}{dx} = x^{x^x}\left(x^x\left[\frac{1}{x} + (1 + \log x) \times \log x\right]\right)$ $\qquad \therefore \frac{dy}{dx} = x^{x^x}\left[x^x\left(\frac{1}{x} + \log x + (\log x)^2\right)\right]$

Q78. We have $y = \log[x + \sqrt{x^2 + a^2}]$ $\qquad \Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \times \left(1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \times \left(\frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}}\right)$ $\qquad \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}}$

$\Rightarrow \sqrt{x^2 + a^2}\,\frac{dy}{dx} = 1$ $\qquad \Rightarrow \sqrt{x^2 + a^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{2x}{2\sqrt{x^2 + a^2}} = 0$ $\qquad \therefore (x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Q79. Given $x = a\cos\theta + b\sin\theta$ and, $y = a\sin\theta - b\cos\theta$

$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$ $\qquad$ and, $\qquad \frac{dy}{d\theta} = a\cos\theta + b\sin\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{a\cos\theta + b\sin\theta}{b\cos\theta - a\sin\theta} = -\frac{x}{y} \ldots$(i)

$\Rightarrow y\frac{dy}{dx} = -x$ $\qquad \Rightarrow y\frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{dy}{dx} = -1$

By (i), we get : $y\frac{d^2y}{dx^2} + \left(-\frac{x}{y}\right) \times \frac{dy}{dx} + 1 = 0$ $\qquad \therefore y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Q80. We have $x = ae^\theta(\sin\theta - \cos\theta)$ and $y = ae^\theta(\sin\theta + \cos\theta)$
On differentiating x and y w.r.t. θ both the sides, we get :

$x = ae^\theta(\sin\theta - \cos\theta) \Rightarrow \frac{dx}{d\theta} = ae^\theta(\sin\theta - \cos\theta) + ae^\theta(\cos\theta + \sin\theta) = 2ae^\theta \sin\theta$

And, $y = ae^\theta(\sin\theta + \cos\theta) \Rightarrow \frac{dy}{d\theta} = ae^\theta(\sin\theta + \cos\theta) + ae^\theta(\cos\theta - \sin\theta) = 2ae^\theta \cos\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2ae^\theta \cos\theta \times \frac{1}{2ae^\theta \sin\theta} = \cot\theta$ $\qquad$ So, $\left.\frac{dy}{dx}\right]_{at\ \theta=\pi/4} = \cot(\pi/4) = 1$.

Q81. We have $y = Pe^{ax} + Qe^{bx}$ $\qquad$ [On dividing both sides by e^{bx}

$\Rightarrow e^{-bx}y = Pe^{(a-b)x} + Q$ $\qquad$ [On diff. w.r.t. x both sides

$\Rightarrow e^{-bx}y' - bye^{-bx} = P(a-b)e^{(a-b)x}$ $\qquad \Rightarrow e^{-bx}(y' - by) = P(a-b)e^{(a-b)x}$

On multiplying both the sides by e^{bx-ax}, we get :

$\Rightarrow e^{-ax}(y' - by) = P(a-b)$ $\qquad$ [Again diff. w.r.t. x both sides

$\Rightarrow e^{-ax}(y'' - by') - ae^{-ax}(y' - by) = 0$ $\qquad \Rightarrow e^{-ax}[y'' - by' - ay' + aby] = 0$

i.e., $y'' - (a+b)y' + aby = 0$ $\qquad$ or, $\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$.

Q82. We have $(\cos x)^y = (\cos y)^x$ $\qquad \Rightarrow \log(\cos x)^y = \log(\cos y)^x$ $\qquad \Rightarrow y\log\cos x = x\log\cos y$

Diff. w.r.t. x both sides, we get : $y\frac{-\sin x}{\cos x} + \log\cos x\frac{dy}{dx} = x\frac{-\sin y}{\cos y}\frac{dy}{dx} + \log\cos y$

$(\log\cos x + x\tan y)\dfrac{dy}{dx} = \log\cos y + y\tan x \quad \therefore \dfrac{dy}{dx} = \dfrac{\log\cos y + y\tan x}{\log\cos x + x\tan y}$

Q83. We have $f(x) = \sqrt{x^2+1}$; $g(x) = \dfrac{x+1}{x^2+1}$ and $h(x) = 2x-3$

$\Rightarrow f'(x) = \dfrac{2x}{2\sqrt{x^2+1}}$; $g'(x) = \dfrac{(x^2+1)-(x+1)2x}{(x^2+1)^2}$ and $h'(x) = 2$

$\Rightarrow f'(x) = \dfrac{x}{\sqrt{x^2+1}}$; $g'(x) = \dfrac{1-2x-x^2}{(x^2+1)^2}$ and $h'(x) = 2$

Now $f'[h'\{g'(x)\}] = f'\left[h'\left\{\dfrac{1-2x-x^2}{(x^2+1)^2}\right\}\right] = f'[2] = \dfrac{2}{\sqrt{2^2+1}} = \dfrac{2}{\sqrt{5}}$.

Q84. We have $y = \sqrt{x+1} - \sqrt{x-1}$ $\quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x+1}} - \dfrac{1}{2\sqrt{x-1}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\sqrt{x-1}-\sqrt{x+1}}{2\sqrt{x+1}\sqrt{x-1}}$ $\quad \Rightarrow 2\sqrt{x^2-1}\dfrac{dy}{dx} = \sqrt{x-1}-\sqrt{x+1} = -y$

$\Rightarrow 4(x^2-1)\left(\dfrac{dy}{dx}\right)^2 = y^2$ $\quad \Rightarrow 4(x^2-1)\times 2\left(\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2} + 8x\left(\dfrac{dy}{dx}\right)^2 = 2y\left(\dfrac{dy}{dx}\right)$

$\Rightarrow (x^2-1)\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right) = \dfrac{y}{4}$ $\quad \therefore (x^2-1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - \dfrac{1}{4}y = 0$.

Q85. Given $y = x^3\log\left(\dfrac{1}{x}\right)$ $\quad \Rightarrow y = -x^3\log x$

$\Rightarrow \dfrac{dy}{dx} = -x^3\times\dfrac{1}{x} - 3x^2\log x = -x^2(1+3\log x)$

$\Rightarrow \dfrac{dy}{dx} = -x^2(1+3\log x)$ $\quad \Rightarrow \dfrac{d^2y}{dx^2} = -x^2\times\dfrac{3}{x} - 2x(1+3\log x)$

$\Rightarrow x\dfrac{d^2y}{dx^2} = -3x^2 - 2x^2(1+3\log x)$ $\quad \Rightarrow x\dfrac{d^2y}{dx^2} + 3x^2 = -2x^2(1+3\log x)$

$\Rightarrow x\dfrac{d^2y}{dx^2} + 3x^2 = 2\dfrac{dy}{dx}$ $\quad \therefore x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3x^2 = 0$.

Q86. Put $x = \cos 2\theta$ in $2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$, we get $2\tan^{-1}\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}} = 2\theta = \cos^{-1}x$.

Hence $y = e^{\sin^2 x}\left\{2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right\}$ $\quad \Rightarrow y = e^{\sin^2 x}\cos^{-1}x$ $\quad \Rightarrow \log y = \log e^{\sin^2 x} + \log\cos^{-1}x$

$\Rightarrow \log y = \sin^2 x + \log\cos^{-1}x$ $\quad \Rightarrow \dfrac{1}{y}\times\dfrac{dy}{dx} = \sin 2x + \dfrac{1}{\cos^{-1}x}\times\dfrac{-1}{\sqrt{1-x^2}}$

$\therefore \dfrac{dy}{dx} = e^{\sin^2 x}\left\{2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right\}\left[\sin 2x - \dfrac{1}{\sqrt{1-x^2}\cos^{-1}x}\right]$.

Q87. As $2x = y^{1/5} + y^{-1/5}$ $\quad \Rightarrow \left(x+\sqrt{x^2-1}\right) + \left(x-\sqrt{x^2-1}\right) = y^{1/5} + y^{-1/5}$

$\Rightarrow \left(x+\sqrt{x^2-1}\right) + \left(x+\sqrt{x^2-1}\right)^{-1} = y^{1/5} + (y^{1/5})^{-1}$

On comparing $y^{1/5} = x + \sqrt{x^2-1}$ $\quad \therefore y = [x+\sqrt{x^2-1}]^5$

$\Rightarrow y_1 = 5[x+\sqrt{x^2-1}]^4 \times \left(1+\frac{2x}{2\sqrt{x^2-1}}\right) = \frac{5y}{\sqrt{x^2-1}}$ $\Rightarrow \sqrt{x^2-1}\times y_1 = 5y$

$\Rightarrow \sqrt{x^2-1}\times y_2 + y_1 \times \frac{2x}{2\sqrt{x^2-1}} = 5y_1$ $\Rightarrow (x^2-1)y_2 + xy_1 = 5\left(\frac{5y}{\sqrt{x^2-1}}\right)\sqrt{x^2-1}$

$\therefore (x^2-1)y_2 + xy_1 = 25y$.