Chapter 03

SOLUTIONS Of EXERCISE FOR PRACTICE

■ Based On Continuity & Differentiability

VERY SHORT ANSWER TYPE QUESTIONS

Q01. Since modulus function is always continuous for all $x \in R$ so, $f(x)$ is continuous.

Q02. Here $f(x)$ is a polynomial function which is continuous for all real numbers.

Q03. This function is defined for all $x \in R - \{0\}$ so, it is continuous everywhere in its domain.

Q04. We know that sin x and cos x both are continuous for all $x \in R$. Therefore $f(x)$ is also continuous for all $x \in R$.

Q05. Since modulus function is always continuous for all $x \in R$ so, $f(x)$ is continuous.

Q06. This function is defined for all $x \in R - \{5\}$ so, it is continuous everywhere in its domain.

Q07. We have $f(x) = \dfrac{1}{\log_e x}$.

For $f(x)$ to be defined, $\log_e x \neq 0,\ x \neq 1$. So domain of $f(x) =\ x \in (0,\infty) - \{1\}$.

$\therefore$ the function is continuous everywhere in its domain $(0,\infty) - \{1\}$.

Q08. Domain of $f(x) = e^x \log|x|$ is $R - \{0\}$ therefore, f is continuous in $x \in (-\infty,\infty) - 0$.

Q09. The function f is defined if $1 - 9x^2 > 0$ and $x > 0$ i.e, $(1-3x)(1+3x) > 0$ and $x > 0$

$\therefore x \in (-1/3, 1/3)$ and $x > 0$. So, domain of $f(x) =\ x \in (0, 1/3)$.

Hence f is continuous in its domain $x \in (0,1/3)$.

Q10. $f(x) = |3x - 5|$ is not differentiable at x = 5/3.

SHORT ANSWER TYPE QUESTIONS

Q01. Right Hand Limit (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$

$$\Rightarrow \quad = \lim_{x\to 0^+} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x} \times \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$\Rightarrow \quad = \lim_{x\to 0^+} \frac{(1+\sin x) - (1-\sin x)}{x} \times \frac{1}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$\Rightarrow \quad = \lim_{x\to 0^+} \frac{2\sin x}{x} \times \frac{1}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} = 2\times 1 \times \frac{1}{\sqrt{1+\sin 0} + \sqrt{1-\sin 0}} = \frac{2}{2} = 1$$

Left Hand Limit (at x = 0) : $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$

$$\Rightarrow \quad = \lim_{x\to 0^-} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x} \times \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$\Rightarrow \quad = \lim_{x\to 0^-} \frac{2\sin x}{x} \times \frac{1}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} = 2\times 1 \times \frac{1}{\sqrt{1+\sin 0} + \sqrt{1-\sin 0}} = \frac{2}{2} = 1$$

Also, $f(0) = 1$.

Since LHL (at x = 0) = RHL (at x = 0) = f (0) so, f (x) is continuous at x = 0.

Q02. Right Hand Limit (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \sin\left(\dfrac{1}{x}\right)$

$$\Rightarrow \quad = 0^2 \sin\left(\frac{1}{0}\right) = 0 \times (\text{a value oscillating between } -1 \text{ and } 1) = 0 \qquad [\because -1 \le \sin\theta \le 1 \ \ \forall\, \theta \in R]$$

Left Hand Limit (at x = 0) : $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \sin\left(\frac{1}{x}\right)$

$\Rightarrow \quad = 0^2 \sin\left(\frac{1}{0}\right) = 0 \times (\text{a value oscillating between } -1 \text{ and } 1) = 0$

Also, f (0) = 0.

Since LHL (at x = 0) = RHL (at x = 0) = f (0) so, f (x) is continuous at x = 0.

Q03. Right Hand Limit (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{x} + \cos x$

$\Rightarrow \quad = 1 + \cos 0 = 1 + 1 = 2$

Left Hand Limit (at x = 0) : $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{x} + \cos x$

$\Rightarrow \quad = 1 + \cos 0 = 1 + 1 = 2$

Also, f (0) = 2.

Since LHL (at x = 0) = RHL (at x = 0) = f (0) so, f (x) is continuous at x = 0.

Q04. We have $f(x) = \begin{cases} \frac{x - |x|}{2}, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases} = \begin{cases} \frac{x - x}{2} = 0, & \text{if } x > 0 \\ 2, & \text{if } x = 0 \\ \frac{x - (-x)}{2} = x, & \text{if } x < 0 \end{cases}$

Right Hand Limit (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 0 = 0$

Also, f (0) = 2.

Since RHL (at x = 0) $\neq$ f (0) so, f (x) is discontinuous at x = 0.

Q05. We have $f(x) = \begin{cases} \frac{2|x| + x^2}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} \frac{-2x + x^2}{x} = x - 2, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \frac{2x + x^2}{x} = x + 2, & \text{if } x > 0 \end{cases}$

Right Hand Limit (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + 2 = 0 + 2 = 2$

Also, f (0) = 0.

Since RHL (at x = 0) $\neq$ f (0) so, f (x) is discontinuous at x = 0.

Q06. We have f (1) = 1 + 5 = 6

Right Hand Limit (at x = 1) : $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x - 5 = 1 - 5 = -4$

Since RHL (at x = 1) $\neq$ f (1) so, f (x) is discontinuous at x = 1.

Q07. Try yourself.

Q08. We have f (0) = 7

Right Hand Limit (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^x - 1}{\log(1+2x)} = \lim_{x\to 0^+} \left(\frac{e^x - 1}{x}\right)\left(\frac{2x}{\log(1+2x)}\right) \times \frac{1}{2}$

$\Rightarrow \quad = 1 \times 1 \times \frac{1}{2} = \frac{1}{2}$ $\quad [\because \text{ when } x \to 0 \Rightarrow 2x \to 0$

Since RHL (at x = 0) $\neq$ f (0) so, f (x) is discontinuous at x = 0.

Q09. We have f (0) = k …(i)

RHL (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1 - \cos 4x}{8x^2} = \lim_{x\to 0^+} \frac{2\sin^2 2x}{8x^2} = \lim_{x\to 0^+} \frac{\sin^2 2x}{4x^2} = 1 \ldots\text{(ii)}$

$[\because x \to 0 \Rightarrow 2x \to 0$

As f is continuous at x = 0, so RHL (at x = 0) = f (0) = LHL (at x = 0)

Therefore by (i) and (ii), we get : k = 1.

Q10. We have f (0) = k …(i)

RHL (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1} = \lim_{x\to 0^+} \frac{\cos 2x - 1}{\sqrt{x^2+1}-1} \times \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}$

$\Rightarrow \quad = \lim_{x\to 0^+} \frac{-(1-\cos 2x)}{(x^2+1)-1} \times \left(\sqrt{x^2+1}+1\right) = \lim_{x\to 0^+} \frac{-2\sin^2 x}{x^2} \times \left(\sqrt{x^2+1}+1\right)$

$\Rightarrow \quad = -2(1)^2\left(\sqrt{0^2+1}+1\right) = -4 \ldots$(ii)

As f is continuous at x = 0, so RHL (at x = 0) = f (0) = LHL (at x = 0)

Therefore by (i) and (ii), we get : k = –4.

Q11. Continuity at x = 3 :

We have f (3) = 1 …(i)

RHL (at x = 3) : $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} ax + b = 3a + b \ldots$(ii)

Since f is continuous at x = 3 therefore by (i) and (ii) we get : 3a + b = 1 …(A)

Continuity at x = 5 :

We have f (5) = 7 …(iii)

LHL (at x = 5) : $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} ax + b = 5a + b \ldots$(iv)

Since f is continuous at x = 5 therefore by (iii) and (iv) we get : 5a + b = 7 …(B)

Now solving (A) and (B), we get : a = 3, b = –8.

Q12. We have f (2) = k …(i)

RHL (at x = 2) : $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{2^{x+2}-16}{4^x - 16} = \lim_{x\to 2^+} \frac{4(2^x-4)}{(2^x-4)(2^x+4)}$

$\Rightarrow \quad = \lim_{x\to 2^+} \frac{4}{(2^x+4)} = \frac{4}{2^2+4} = \frac{1}{2} \ldots$(ii)

Since f is continuous at x = 2 therefore by (i) and (ii), k = 1/2.

Q13. We have f (4) = a + b …(i)

RHL (at x = 4) : $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{x-4}{|x-4|} + b = \lim_{x\to 4^+} \frac{x-4}{x-4} + b = 1 + b \ldots$(ii)

LHL (at x = 4) : $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{x-4}{|x-4|} + a = \lim_{x\to 4^-} \frac{x-4}{-(x-4)} + a = -1 + a \ldots$(iii)

Since f (x) is continuous at x = 4, so $\lim_{x\to 4} f(x) = f(4)$.

Therefore solving (i), (ii) and (iii) we get : a = 1, b = –1.

Q14. We have f (1) = 11 …(i)

RHL (at x = 1) : $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3ax + b = 3a \times 1 + b = 3a + b \ldots$(ii)

LHL (at x = 1) : $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 5ax - 2b = 5a - 2b \ldots$(iii)

Since f (x) is continuous at x = 1, so $\lim_{x\to 1} f(x) = f(1)$.

Therefore solving (i), (ii) and (iii) we get : a = 3, b = 2.

Q15. We have f (2) = a …(i)

RHL (at x = 2) : $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+1) = 2 + 1 = 3 \ldots$(ii)

Since f (x) is continuous at x = 2, so $\lim_{x\to 2^+} f(x) = f(2)$.

Therefore by (i) and (ii), we get : a = 3.

Q16. We have f (3) = 3a + 1 …(i)

RHL (at x = 3) : $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} bx+3 = 3b+3$ …(ii)

Since f (x) is continuous at x = 3, so $\lim_{x\to 3^+} f(x) = f(3)$.

Therefore by (i) and (ii), we get : 3a + 1 = 3b + 3 $\Rightarrow 3a - 3b = 2$.

Q17. We have $f(\pi) = k\pi + 1$ …(i)

RHL (at x = π) : $\lim_{x\to \pi^+} f(x) = \lim_{x\to \pi^+} \cos x = \cos\pi = -1$ …(ii)

Since f (x) is continuous at x = π, so $\lim_{x\to \pi^+} f(x) = f(\pi)$.

Therefore by (i) and (ii), we get : $k\pi + 1 = -1$ $\Rightarrow k\pi = -2$ $\therefore k = -\frac{2}{\pi}$.

Q18. We've f (2) = a(2)^2 = 4a …(i)

RHL (at x = 2) : $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3 = 3$ …(ii)

Since f (x) is continuous at x = 2, so $\lim_{x\to 2^+} f(x) = f(2)$.

Therefore by (i) and (ii), we get : $4a = 3$ $\Rightarrow a = 3/4$.

Q19. Let $f(x) = \left|1 - x + |x|\right|$.

Also let $g(x) = |x|$, $h(x) = 1 - x + |x|$. Both of these functions g (x) and h (x) are continuous for all x ∈ R. Therefore, f (x) = (g o h) (x) = g (h (x)).

Since it is known that for real valued functions g and h, such that (g o h) is defined at c, if h is continuous at c and if g is continuous at h (c), then (g o h) is continuous at c.

Therefore f (x) = $\left|1 - x + |x|\right|$ is continuous function for all real value of x.

Q20. **(a)** Let g (x) = |x – 1| which is a modulus function and so, it is continuous ∀ x ∈ R

Also, let h (x) = |x + 1|, which is also a modulus function and so, it is also continuous ∀ x ∈ R.

$\because (g+h)(x) = g(x) + h(x) = |x-1| + |x+1| = f(x)$

$\therefore f(x)$ is also continuous function ∀ x ∈ R.

(b) We've $f(x) = |x-1| + |x+1| = \begin{cases} -(x-1)-(x+1) = -2x, & x < -1 \\ -(x-1)+(x+1) = 2, & -1 \le x < 1 \\ x-1+x+1 = 2x, & x \ge 1 \end{cases}$

Differentiability at x = –1 :

$$Lf'(-1) = \lim_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1^-} \frac{-2x-2}{x+1} = \lim_{x\to -1^-} \frac{-2(x+1)}{x+1} = -2$$

$$Rf'(-1) = \lim_{x\to -1^+} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1^+} \frac{2-2}{x+1} = \lim_{x\to -1^+} \frac{0}{x+1} = 0 \ne Lf'(-1)$$

Hence f (x) isn't differentiable at x = –1.

Differentiability at x = 1 :

$$Lf'(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{2-2}{x+1} = \lim_{x\to 1^-} \frac{0}{x+1} = 0$$

$$Rf'(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{2x-2}{x-1} = \lim_{x\to 1^+} \frac{2(x-1)}{x-1} = \lim_{x\to 1^+} 2 = 2 \ne Lf'(1)$$

Hence f (x) isn't differentiable at x = 1 as well.

Q21. We have f (0) = 3 (0) – 2 = –2 …(i)

RHL (at x = 0) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+1 = 0+1 = 1$

LHL (at x = 0) : $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x - 2 = 3\times 0 - 2 = -2$

As RHL (at x = 0) $\neq$ LHL (at x = 0) = f (0) so, f is discontinuous at x = 0.

Q22. We have f (5) = λ …(i)

RHL (at x = 5) : $\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{x^2-25}{x-5} = \lim_{x\to 5^+} \frac{(x-5)(x+5)}{x-5} = \lim_{x\to 5^+}(x+5) = 5+5 = 10$ …(ii)

Since f is continuous at x = 5 so, f (5) = RHL (at x = 5)

Therefore by (i) and (ii), we get : $\lambda = 10$.

Q23. We've f (0) = 3/2

LHL (at x = 0) : $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin 3x}{\tan 2x} = \lim_{x\to 0^-}\left(\frac{\sin 3x}{3x}\right)\left(\frac{2x}{\tan 2x}\right)\frac{3}{2} = \frac{3}{2}$ $\left[\begin{array}{l}\because x\to 0\\ \therefore 2x\to 0, 3x\to 0\end{array}\right.$

RHL (at x = 0) : $\lim_{x\to 0^+} \frac{\log(1+3x)}{e^{2x}-1} = \lim_{x\to 0^+}\left[\frac{\log(1+3x)}{3x}\right]\left[\frac{2x}{e^{2x}-1}\right]\times\frac{3}{2} = \frac{3}{2}$ $\left[\begin{array}{l}\because x\to 0\\ \therefore 2x\to 0, 3x\to 0\end{array}\right.$

Since $f(0) = \lim_{x\to 0} f(x)$, therefore f is continuous at x = 0.

Q24. We've f (1) = k $(1)^2$ = k …(i)

LHL (at x = 0) : $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 4 = 4$ …(ii)

As f is continuous at x = 1 so, by (i) and (ii) we get : k = 4.

Q25. Since f is continuous on [0, 8] so, it is continuous on all the points belonging to [0, 8].

Continuity at x = 2 :

We have f (2) = 2^2 + 2a + b = 2a + b + 4 …(i)

RHL (at x = 2) : $\lim_{x\to 2^+} 3x + 2 = 3\times 2 + 2 = 8$ …(ii)

As f is continuous at x = 2 so, by (i) and (ii) we get : 2a + b = 4 …(A)

Continuity at x = 4 :

We have f (4) = 3(4) + 2 = 14 …(iii)

RHL (at x = 4) : $\lim_{x\to 4^+}(2ax + 5b) = 2a(4) + 5b = 8a + 5b$ …(iv)

As f is continuous at x = 4 so, by (iii) and (iv) we get : 8a + 5b = 14 …(B)

Solving (A) and (B), we get : a = 3, b = –2.

Q26. We've f (0) = log k …(i)

RHL (at x = 0) : $\lim_{x\to 0^+} \frac{x(3^x-1)}{1-\cos x} = \lim_{x\to 0^+} x^2\left(\frac{3^x-1}{x}\right)\left(\frac{\frac{x^2}{4}}{2\sin^2\frac{x}{2}}\right)\frac{4}{x^2} = \lim_{x\to 0^+}\left(\frac{3^x-1}{x}\right)\left(\frac{\frac{x^2}{4}}{\sin^2\frac{x}{2}}\right)\times 2$

$\Rightarrow \quad = 2\log 3$ …(ii)

As f is continuous at x = 0 so, by (i) and (ii), we get : $\log k = 2\log 3 \quad \Rightarrow k = 9$.

Q27. We have f (1) = k …(i)

RHL (at x = 1) : $\lim_{x\to 1^+}(x-1)\tan\frac{\pi x}{2} = \lim_{h\to 0} h\tan\frac{\pi(1+h)}{2}$ $\left[\begin{array}{l}\text{Put } x = 1+h\\ \text{As } x\to 1 \Rightarrow h\to 0\end{array}\right.$

$\Rightarrow \quad = \lim_{h\to 0} h\tan\left(\frac{\pi}{2}+\frac{\pi h}{2}\right) = -\lim_{h\to 0} h\cot\left(\frac{\pi h}{2}\right)$

$\Rightarrow \quad = -\lim_{h\to 0}\frac{\frac{\pi h}{2}}{\tan\left(\frac{\pi h}{2}\right)}\times\frac{2}{\pi h}\times h$

$$\Rightarrow \quad = -\lim_{h\to 0} \frac{\frac{\pi h}{2}}{\tan\left(\frac{\pi h}{2}\right)} \times \frac{2}{\pi} = -1 \times \frac{2}{\pi} = -\frac{2}{\pi} \ldots\text{(ii)} \qquad [\because h \to 0 \Rightarrow \pi h/2 \to 0$$

As f (x) is continuous at x = 1 so, by (i) and (ii) we get : $k = -\frac{2}{\pi}$.

Q28. Since f (x) is continuous on $[0,\pi]$ so, it is continuous at all $x \in [0,\pi]$.

Continuity at $x = \pi/4$:

We have $f(\pi/4) = 2(\pi/4)\cot(\pi/4) + b = b + \pi/2 \ldots\text{(i)}$

LHL (at $x = \pi/4$) : $\lim_{x\to\left(\frac{\pi}{4}\right)^-} x + a\sqrt{2}\sin x = \frac{\pi}{4} + a\sqrt{2}\sin\frac{\pi}{4} = \frac{\pi}{4} + a \ldots\text{(ii)}$

As f is continuous at $x = \pi/4$ so, by (i) and (ii) we get : $b + \frac{\pi}{2} = \frac{\pi}{4} + a \quad \Rightarrow a - b = \frac{\pi}{4} \ldots\text{(A)}$

Continuity at $x = \pi/2$:

We have $f(\pi/2) = a\cos\pi - b\sin\frac{\pi}{2} = -a - b \ldots\text{(iii)}$

LHL (at $x = \pi/2$) : $\lim_{x\to\left(\frac{\pi}{2}\right)^-} 2x\cot x + b = 2\times\frac{\pi}{2}\cot\frac{\pi}{2} + b = 0 + b = b \ldots\text{(iv)}$

As f is continuous at $x = \pi/2$ so, by (iii) and (iv) we get :

$-a - b = b \qquad \Rightarrow b = -\frac{a}{2} \ldots\text{(B)}$

Solving (A) and (B) we get : $a = \frac{\pi}{6}, b = -\frac{\pi}{12}$.

Q29. The function f is defined for all real number and it can be expressed as the composition of two functions f = g o h, where g (x) = sin x and h (x) = x^2.
[$\because$ (g o h) (x) = g (h (x)) = g (x^2) = sin x^2 = f (x)]
So, we need to prove that g (x) and h (x) are continuous functions.

We have g (x) = sin x. Let c be any real number.
Then g (c) = sin c …(i) [As sin x is defined for every real number]
LHL (at x = c): $\lim_{x\to c^-} g(x) = \lim_{x\to c^-} \sin x$ [Put $x = c - h$ so that as $x \to c$, $h \to 0$

$= \lim_{h\to 0} \sin(c - h) = \sin(c - 0) = \sin c \quad \ldots\text{(ii)}$

RHL (at x = c): $\lim_{x\to c^+} g(x) = \lim_{x\to c^+} \sin x$ [Put $x = c + h$ so that as $x \to c$, $h \to 0$

$= \lim_{h\to 0} \sin(c + h) = \sin(c + 0) = \sin c \quad \ldots\text{(iii)}$

By (i), (ii) & (iii), it is clearly evident that LHL (at x = c) = RHL (at x = c) = f (c).
So, g (x) is continuous at all real values of x.

Also, we have h (x) = x^2.

Clearly, the function h is defined for every real number.
Let k be a real number, then h (k) = k^2.
And $\lim_{x\to k} h(x) = \lim_{x\to k} x^2 = k^2$.

$\because \lim_{x\to k} h(x) = h(k) \quad \Rightarrow \quad h(x)$ is a continuous function.

Since it is known that for real valued functions g and h, such that (g o h) is defined at c, if h is continuous at c and if g is continuous at h (c), then (g o h) is continuous at c.
Therefore f (x) = (g o h) (x) = sin (x^2) is continuous function.

Q30. We have $f(0) = \dfrac{b^2 - a^2}{2} \ldots$(i)

RHL (at x = 0) : $\lim_{x\to 0^+} \dfrac{\cos ax - \cos bx}{x^2} = \lim_{x\to 0^+} \dfrac{(\cos ax - 1) + (1 - \cos bx)}{x^2}$

$$\Rightarrow \quad = \lim_{x\to 0^+} \frac{-2\sin^2 \frac{ax}{2} + 2\sin^2 \frac{bx}{2}}{x^2} = 2\lim_{x\to 0^+}\left(\left(\frac{\sin^2 \frac{bx}{2}}{\left(\frac{bx}{2}\right)^2}\right)\frac{b^2}{4} - \left(\frac{\sin^2 \frac{ax}{2}}{\left(\frac{ax}{2}\right)^2}\right)\frac{a^2}{4}\right)$$

$$\Rightarrow \quad = 2\left[\frac{b^2}{4} - \frac{a^2}{4}\right] = \frac{b^2 - a^2}{2} \qquad \left[\begin{array}{l}\because x \to 0 \\ \therefore ax/2 \to 0,\ bx/2 \to 0\end{array}\right.$$

LHL (at x = 0) : $\lim_{x\to 0^-} \dfrac{\cos ax - \cos bx}{x^2} = \lim_{x\to 0^-} \dfrac{-2\sin \frac{ax+bx}{2} \sin \frac{ax-bx}{2}}{x^2}$

$$\Rightarrow \quad = -2\lim_{x\to 0^-} \frac{\sin x\left(\frac{a+b}{2}\right)}{x\left(\frac{a+b}{2}\right)}\left[\frac{x^2\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)}{x^2}\right]\frac{\sin x\left(\frac{a-b}{2}\right)}{x\left(\frac{a-b}{2}\right)}$$

$$\Rightarrow \quad = -2\lim_{x\to 0^-} \frac{\sin x\left(\frac{a+b}{2}\right)}{x\left(\frac{a+b}{2}\right)}\left[\frac{a^2 - b^2}{4}\right]\frac{\sin x\left(\frac{a-b}{2}\right)}{x\left(\frac{a-b}{2}\right)} = -2\left(\frac{a^2 - b^2}{4}\right) = \left(\frac{b^2 - a^2}{2}\right)$$

$$\left[\begin{array}{l}\because x \to 0 \\ \therefore x\left(\frac{a+b}{2}\right) \to 0,\ x\left(\frac{a-b}{2}\right) \to 0\end{array}\right.$$

Since $f(0) = \lim_{x\to 0} f(x)$ so, f (x) is continuous at x = 0.

Q31. We have f (0) = 2 …(i)

LHL (at x = 0) : $\lim_{x\to 0^-} \dfrac{\sin x}{x} = 1$ …(ii) and, RHL (at x = 0) : $\lim_{x\to 0^+} \dfrac{\sin x}{x} = 1$…(iii)

By (i), (ii) & (iii), we can see that $f(0) \neq \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ so, the function f (x) is discontinuous at x = 0.
In order to make it continuous at x = 0, the value of f (x) at x = 0 should be 1.

Q32. We've f (0) = a

LHL (at x = 0) : $\lim_{x\to 0^-} \dfrac{1 - \cos 4x}{x^2} = \lim_{x\to 0^-} \dfrac{2\sin^2 2x}{4x^2} \times 4 = 1^2 \times 8 = 8 \qquad [\because x \to 0 \ \therefore 2x \to 0$

RHL (at x = 0) : $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4} = \lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4} \times \dfrac{\sqrt{16+\sqrt{x}} + 4}{\sqrt{16+\sqrt{x}} + 4}$

$\Rightarrow \quad = \lim_{x\to 0^+} \sqrt{x} \times \frac{\sqrt{16+\sqrt{x}}+4}{16+\sqrt{x}-16} = \lim_{x\to 0^+} \sqrt{16+\sqrt{x}}+4 = \sqrt{16+\sqrt{0}}+4 = 8$

As f is continuous at x = 0 then, $f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ $\quad \therefore a = 8$.

Q33. We have $f(x) = \begin{cases} \frac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ $\quad \Rightarrow f(x) = \begin{cases} -\frac{\sin x}{x}, & x < 0 \\ 1, & x = 0 \\ \frac{\sin x}{x}, & x > 0 \end{cases}$

As f (0) = 1.

LHL (at x = 0) : $\lim_{x\to 0^-} -\frac{\sin x}{x} = -1$ and, RHL (at x = 0) : $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$

Since f (0) = RHL (at x = 0) ≠ LHL (at x = 0) so, f is discontinuous at x = 0.

Q34. We have $f(0) = \lambda \ldots$(i)

LHL (at x = 0) : $\lim_{x\to 0^-} \frac{3x-\tan x}{5x-\sin x} = \lim_{x\to 0^-} \frac{3-\frac{\tan x}{x}}{5-\frac{\sin x}{x}} = \frac{3-1}{5-1} = \frac{2}{4} = \frac{1}{2} \ldots$(ii)

RHL (at x = 0) : $\lim_{x\to 0^+} 3x^2 - 4x + \frac{1}{2} = 3(0)^2 - 4(0) + \frac{1}{2} = \frac{1}{2} \ldots$(iii)

Since f is continuous at x = 0 so, by (i), (ii) and (iii) we get : $\lambda = 1/2$.

Q35. Given $f(x) = |x| = \begin{cases} -x, & x < 0 \\ 0, & x = 0 \\ x, & x > 0 \end{cases}$

We have f (0) = 0.

Now LHD (at x = 0) : $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{-x-0}{x} = \lim_{x\to 0^-}(-1) = -1$

And, RHD (at x = 0) : $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x-0}{x} = \lim_{x\to 0^+}(1) = 1$

$\because Lf'(0) \neq Rf'(0)$ so, the function f (x) is not differentiable at x = 0.

Q36. We have $f(x) = |x-1| = \begin{cases} -(x-1), & x < 1 \\ x-1, & x \geq 1 \end{cases}$

$\therefore f(1) = 1-1 = 0$

Now $Lf'(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{-x+1-0}{x-1} = \lim_{x\to 1^-}(-1) = -1$

And, $Rf'(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{x-1-0}{x-1} = \lim_{x\to 1^+}(1) = 1$

$\because Lf'(1) \neq Rf'(1)$ so, the function f (x) is not differentiable at x = 1.

Q37. We have $f(x) = |x-3| = \begin{cases} x-3, & \text{if } x \geq 3 \\ 3-x, & \text{if } x < 3 \end{cases}$.

Continuity at x = 3 :

LHL (at x = 3) : $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(3-x) = 3-3 = 0$

RHL (at x = 3) : $\lim_{x\to 3^+}(x-3) = 3-3 = 0$.

And, $f(3) = (3-3) = 0$

As LHL (at x = 3) = RHL (at x = 3) = f (3)
So, f (x) is continuous at x = 3.

Differentiability at x = 3 :

LHD (at x = 3) : $\lim_{x\to3^-}\frac{f(x)-f(3)}{x-3}=\lim_{x\to3^-}\frac{(3-x)-0}{x-3}=\lim_{x\to3^-}(-1)=-1$

RHD (at x = 3) : $\lim_{x\to3^+}\frac{f(x)-f(3)}{x-3}=\lim_{x\to3^+}\frac{(x-3)-0}{x-3}=\lim_{x\to3^+}(1)=1$.

As LHD (at x = 3) $\neq$ RHD (at x = 3)
So, f (x) is not differentiable at x = 3.

Q38. We have $f(x)=x\,|\,x\,|=\begin{cases}x(x)=x^2, \text{ if } x\geq 0\\ x(-x)=-x^2, \text{ if } x<0\end{cases}$

Now f (0) = 0^2 = 0

$Lf'(0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{-x^2-0}{x}=\lim_{x\to0^-}(-x)=-0=0$

And, $Rf'(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{x^2-0}{x}=\lim_{x\to0^+}(x)=0$

$\because Lf'(0)=Rf'(0)$ so, the function f (x) is differentiable at x = 0.

Q39. It can be easily shown that |x – 1| is not differentiable at x = 1 and similarly, |x – 2| is not differentiable at x = 2. Therefore $f(x)=|x-1|+|x-2|$ is not differentiable.

Q40. As f is differentiable at x = 1 so, it is continuous at x = 1 as well.
Now f (1) = 1^2 + 3(1) + a = 4 + a …(i)
RHL (at x = 1) : $\lim_{x\to1^+}bx+2=b\times1+2=b+2$ …(ii)
By (i) & (ii), b = a + 2 …(A)

Also, $Lf'(1)=\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^-}\frac{x^2+3x+a-(4+a)}{x-1}=\lim_{x\to1^-}\frac{(x-1)(x+4)}{x-1}$

$\Rightarrow \quad =\lim_{x\to1^-}(x+4)=1+4=5$

And, $Rf'(1)=\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^+}\frac{bx+2-(4+a)}{x-1}=\lim_{x\to1^+}\frac{bx-a-2}{x-1}=\lim_{x\to1^+}\frac{bx-b}{x-1}$ [by (A)

$\Rightarrow \quad =\lim_{x\to1^+}\frac{b(x-1)}{x-1}=\lim_{x\to1^+}b=b$

As f is differentiable at x = 1 so, $Rf'(1)=Lf'(1)$ $\quad\therefore b=5$ and, by using (A), a = 3.

Q41. Continuity at x = 0 : We've $f(0)=0$.

RHL (at x = 0) : $\lim_{x\to0^+}x^p\sin\left(\frac{1}{x}\right)$ Put $x=0+h$ As $x\to0,\ h\to0$

$=\lim_{h\to0}(0+h)^p\sin\left(\frac{1}{0+h}\right)=\lim_{h\to0}h^p\sin\left(\frac{1}{h}\right)=0$, when $p>0$

And, LHL (at x = 0) : $\lim_{x\to0^-}x^p\sin\left(\frac{1}{x}\right)$ Put $x=0-h$ As $x\to0,\ h\to0$

$=\lim_{h\to0}(0-h)^p\sin\left(\frac{1}{0-h}\right)=\lim_{h\to0}(-h)^p\sin\left(-\frac{1}{h}\right)=-\lim_{h\to0}(-h)^p\sin\left(\frac{1}{h}\right)=0$, when $p>0$

So, f is continuous at x = 0 when p > 0.

Also, $Lf'(0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{h\to0}\frac{f(-h)-(0)}{-h}=\lim_{h\to0}\frac{(-h)^p\sin(-1/h)}{-h}$ $\left[\begin{array}{l}\text{Put } x=0-h\\ \text{As } x\to0,\ h\to0\end{array}\right.$

$= \lim_{h\to 0}(-h)^{p-1}\sin(-1/h) = 0$, when $p-1>0$ i.e., $p>1$

And, $Rf'(0) = \lim_{x\to 0+}\dfrac{f(x)-f(0)}{x-0} = \lim_{h\to 0}\dfrac{f(h)-(0)}{h} = \lim_{h\to 0}\dfrac{(h)^p\sin(1/h)}{h}$ $\left[\begin{array}{l}\text{Put } x = 0+h \\ \text{As } x\to 0,\ h\to 0\end{array}\right.$

$= \lim_{h\to 0}(h)^{p-1}\sin(1/h) = 0$, when $p-1>0$ i.e., $p>1$

$\therefore$ f is differentiable when $p>1$, hence f (x) is not differentiable when $p\le 1$.

Therefore, function f (x) is continuous but not differentiable when $0<p\le 1$ i.e., $p\in(0,1]$.

Q42. We have f (x) = [x], 0 < x < 3.

As a function is differentiable at a point x = m in its domain if LHD and RHD at x = m are both finite and equal to each other.

Differentiability at x = 1 :

LHD (at x = 1) : $\lim_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1} = \lim_{h\to 0}\dfrac{f(1-h)-f(1)}{(1-h)-1}$ [Put $x = 1-h$ so that as $x\to 1$, $h\to 0$

$\Rightarrow \quad = \lim_{h\to 0}\dfrac{[1-h]-[1]}{-h} = \lim_{h\to 0}\dfrac{0-1}{-h} = \lim_{h\to 0}\dfrac{1}{h} = \dfrac{1}{0} = \infty$.

RHD (at x = 1) : $\lim_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1} = \lim_{h\to 0}\dfrac{f(1+h)-f(1)}{(1+h)-1}$ [Put $x = 1+h$ so that as $x\to 1$, $h\to 0$

$\Rightarrow \quad = \lim_{h\to 0}\dfrac{[1+h]-[1]}{h} = \lim_{h\to 0}\dfrac{1-1}{h} = \lim_{h\to 0}\dfrac{0}{h} = \lim_{h\to 0}(0) = 0$.

Since LHD (at x = 1) ≠ RHD (at x = 1) so, f is not differentiable at x = 1.

[**NOTE** that here we didn't need to evaluate RHD as LHD is already not defined (i.e., ∞), which is an enough reason for f to not be differentiable at x = 1.]

Differentiability at x = 2 :

LHD (at x = 2) : $\lim_{x\to 2^-}\dfrac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-}\dfrac{[x]-[2]}{x-2}$

$\Rightarrow \quad = \lim_{x\to 2^-}\dfrac{1-2}{x-2} = \lim_{x\to 2^-}\dfrac{-1}{x-2} = \dfrac{-1}{0} = -\infty$.

Since LHD (at x = 2) doesn't exist so, f is not differentiable at x = 2.

Q43. Continuity at x = 2 :

We have f (2) = (1 – 2) (2 – 2) = 0

LHL (at x = 2) : $\lim_{x\to 2^-}(1-x)(2-x) = (1-2)(2-2) = 0$

RHL (at x = 2) : $\lim_{x\to 2^+}(3-x) = (3-2) = 1$

Since f (2) = LHL (at x = 2) ≠ RHL (at x = 2) so, f is discontinuous at x = 2.

Differentiability at x = 2 :

RHD (at x = 2) : $\lim_{x\to 2^+}\dfrac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+}\dfrac{(3-x)-0}{x-2} = \dfrac{(3-2)}{2-2} = \dfrac{1}{0} = \infty$.

Since RHD (at x = 2) is not defined, therefore f is not differentiable at x = 2.

Q44. We have f (x) = [x].

<u>Case I</u>: Let c be a real number which is not equal to any integer. It is evident that for all real numbers close to c the value of the function is equal to [c]; i.e., $\lim_{x\to c}f(x) = \lim_{x\to c}[x] = [c]$.

Also f (c) = [c] and hence the greatest integer function is continuous at all non-integral real numbers (since $\lim_{x\to c}f(x) = f(c) = [c]$).

<u>Case II</u>: Let c be an integer.

Then $\lim_{x\to c^-}[x] = c-1$ and $\lim_{x\to c^+}[x] = c$.

$\because \lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$, therefore the greatest integer function is discontinuous at all the integral points.

Q45. We have f (0) = 0

LHL (at x = 0) : $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-}(-1) = -1$

RHL (at x = 0) : $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+}(1) = 1$

$\because$ LHL (at x = 0) $\neq$ RHL (at x = 0) $\neq$ f (0) so, signum function is discontinuous at x = 0.

Q46. Continuity at x = 2 :

We have f (2) = 5.

RHL (at x = 2) : $\lim_{x\to 2^+} ax + b = 2a + b$

$\because$ f is continuous function, so 2a + b = 5 …(i)

Continuity at x = 10 :

We have f (10) = 21.

LHL (at x = 10) : $\lim_{x\to 10^-} ax + b = 10a + b$

$\because$ f is a continuous function, so 10a + b = 21 …(ii)

Solving (i) and (ii), we get : a = 2, b = 1.

Q47. Continuity at x = 2 :

We have f (2) = $2(2)^2 - 2 = 6$.

RHL (at x = 2) : $\lim_{x\to 2^+} 5x - 4 = 5(2) - 4 = 6$

LHL (at x = 2) : $\lim_{x\to 2^-} 2x^2 - x = 2(2)^2 - 2 = 6$

$\because \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$ so, f (x) is continuous at $x = 2$.

Differentiability at $x = 2$:

LHD at $x=2$: $\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{2x^2 - x - 6}{x-2} = \lim_{x\to 2^-} \frac{(2x+3)(x-2)}{x-2} = \lim_{x\to 2^-}(2x+3) = 7$

And, RHD at $x=2$: $\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{5x-4-6}{x-2} = \lim_{x\to 2^+} \frac{5(x-2)}{x-2} = \lim_{x\to 2^+} 5 = 5$.

Since (LHD at $x=2$) $\neq$ (RHD at $x=2$), so f (x) is not differentiable at x = 2.

Hence the given function f (x) is not differentiable at x = 2.

Q48. We take LHD at $x=2$: $\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{(x-1)-(2(2)-3)}{x-2} = \lim_{x\to 2^-} \frac{x-2}{x-2} = \lim_{x\to 2^-}(1) = 1$

And, RHD at $x=2$: $\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{(2x-3)-(2(2)-3)}{x-2} = \lim_{x\to 2^+} \frac{2(x-2)}{x-2} = \lim_{x\to 2^+}(2) = 2$.

Since (LHD at $x=2$) $\neq$ (RHD at $x=2$), so f (x) is not differentiable at x = 2.

Hence the given function f (x) is not differentiable at x = 2.

Q49. We have f (0) = q …(i)

LHL (at x = 0) : $\lim_{x\to 0^-} \frac{\sin(p+1)x + \sin x}{x} = \lim_{x\to 0^-}\left(\frac{\sin(p+1)x}{(p+1)x}\right)(p+1) + \frac{\sin x}{x}$

$\Rightarrow \quad = (p+1) + 1 = p + 2$ …(ii) $\quad [\because x \to 0 \Rightarrow (p+1)x \to 0$

RHL (at x = 0) : $\lim_{x\to 0^+} \frac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}} = \lim_{x\to 0^+} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0^+} \frac{\sqrt{1+x} - 1}{x} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$\Rightarrow \quad = \lim_{x\to 0^+} \frac{1+x-1}{x} \times \frac{1}{\sqrt{1+x}+1} = \lim_{x\to 0^+} \frac{1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$ …(iii)

As function is continuous for all x in R so, by (i), (ii) and (iii) we get : p = –3/2, q = 1/2.

Q50. We have f (1) = m – 1

LHL (at x = 1) : $\lim_{x\to1^-}\frac{1-x^m}{1-x}=\lim_{x\to1^-}\frac{x^m-1^m}{x-1}=m\times1^{m-1}=m$

Since $f(1)\neq\lim_{x\to1^-}f(x)$ so, the point of discontinuity is x = 1, which is the dangerous point.

The driver should not pass this point because life is precious so vehicles should be driven carefully.

Q51. We have f (2) = 2a – 33…(i)

LHL (at x = 2) : $\lim_{x\to2^-}\frac{3x^2-5x-2}{x-2}=\lim_{x\to2^-}\frac{(3x+1)(x-2)}{x-2}=\lim_{x\to2^-}(3x+1)=3\times2+1=7$ …(ii)

Since f is continuous at x = 2 so, by (i) & (ii) we get : 2a – 33 = 7 $\Rightarrow$ a = 20 .

Number of students who participated in the inter school competition = 5(20) + 7 = 107.

Q52. We have $f(0)=\frac{2\times0+1}{0-2}=-\frac{1}{2}$

LHL (at x = 0) : $\lim_{x\to0^-}\frac{\sqrt{1+px}-\sqrt{1-px}}{x}=\lim_{x\to0^-}\frac{\sqrt{1+px}-\sqrt{1-px}}{x}\times\frac{\sqrt{1+px}+\sqrt{1-px}}{\sqrt{1+px}+\sqrt{1-px}}$

$\Rightarrow\quad=\lim_{x\to0^-}\frac{1+px-(1-px)}{x}\times\frac{1}{\sqrt{1+px}+\sqrt{1-px}}=2p\lim_{x\to0^-}\frac{1}{\sqrt{1+px}+\sqrt{1-px}}=\frac{2p}{2}=p$

As f is continuous at x = 0 so, f (0) = LHL (at x = 0) = RHL (at x = 0) so, $p=-\frac{1}{2}$.

Q53. Continuity at x = 1 :

We have $f(1)=|1-1|+|1+2|=3$

LHL (at x = 1) : $\lim_{x\to1^-}|x-1|+|x+2|=|1-1|+|1+2|=3$

RHL (at x = 1) : $\lim_{x\to1^+}|x-1|+|x+2|=|1-1|+|1+2|=3$

$\because f(1)=\lim_{x\to1}f(x),\ \therefore f(x)$ is continuous at x = 1.

Differentiability at x = –2 :

LHD (at x = –2) : $\lim_{x\to-2^-}\frac{f(x)-f(-2)}{x-(-2)}=\lim_{x\to-2^-}\frac{[-(x-1)-(x+2)]-[|-2-1|+|-2+2|]}{x+2}$

$\Rightarrow\quad=\lim_{x\to-2^-}\frac{-2x-1-3}{x+2}=\lim_{x\to-2^-}\frac{-2(x+2)}{x+2}=-2$

RHD (at x = –2) : $\lim_{x\to-2^+}\frac{f(x)-f(-2)}{x-(-2)}=\lim_{x\to-2^+}\frac{[-(x-1)+(x+2)]-3}{x+2}=\lim_{x\to-2^+}\frac{0}{x+2}=0$

Since LHD (at x = –2) ≠ RHD (at x = –2) so, f is not differentiable at x = –2.
Hence the function f is continuous at x = 1 but fails to be differentiable at x = –2 .

Q54. We have f (2) = k …(i)

LHL (at x = 2) : $\lim_{x\to2^-}\frac{\sqrt{5x+2}-\sqrt{4x+4}}{x-2}=\lim_{x\to2^-}\frac{\sqrt{5x+2}-\sqrt{4x+4}}{x-2}\times\frac{\sqrt{5x+2}+\sqrt{4x+4}}{\sqrt{5x+2}+\sqrt{4x+4}}$

$\Rightarrow\quad=\lim_{x\to2^-}\frac{(5x+2)-(4x+4)}{x-2}\times\frac{1}{\sqrt{5x+2}+\sqrt{4x+4}}=\lim_{x\to2^-}\frac{x-2}{x-2}\times\frac{1}{\sqrt{5x+2}+\sqrt{4x+4}}$

$\Rightarrow\quad=\lim_{x\to2^-}\frac{1}{\sqrt{5x+2}+\sqrt{4x+4}}=\frac{1}{\sqrt{12}+\sqrt{12}}=\frac{1}{4\sqrt3}$ …(ii)

As f is continuous at x = 2 so by (i) and (ii), $k=\frac{1}{4\sqrt3}$.

Q55. LHL (at x = π/4) : $\lim\limits_{x\to\frac{\pi}{4}^-} \dfrac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$ Put $x=\frac{\pi}{4}-h \because x\to\frac{\pi}{4}, h\to 0$

$$\therefore \lim_{h\to0}\frac{\tan h}{\cot 2\left(\frac{\pi}{4}-h\right)}=\lim_{h\to0}\frac{\tan h}{\cot\left(\frac{\pi}{2}-2h\right)}=\lim_{h\to0}\frac{\tan h}{\tan 2h}=\lim_{h\to0}\frac{\tan h}{\left(\frac{2\tan h}{1-\tan^2 h}\right)}$$

$$\Rightarrow \quad =\lim_{h\to0}(\tan h)\times\frac{1-\tan^2 h}{2\tan h}=\lim_{h\to0}\frac{1-\tan^2 h}{2}=\frac{1-0}{2}=\frac{1}{2}$$

RHL (at x = π/4) : $\lim\limits_{x\to\frac{\pi}{4}^+} \dfrac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$ Put $x=\frac{\pi}{4}+h \because x\to\frac{\pi}{4}, h\to 0$

$$\therefore \lim_{h\to0}\frac{\tan(-h)}{\cot\left(\frac{\pi}{2}+2h\right)}=\lim_{h\to0}\frac{-\tan h}{-\tan 2h}=\lim_{h\to0}\frac{\tan h}{2\tan h}\times[1-\tan^2 h]=\lim_{h\to0}\frac{1}{2}\times[1-\tan^2 h]=\frac{1}{2}$$

$\because$ LHL $\left(\text{at } x=\frac{\pi}{4}\right)$ = RHL $\left(\text{at } x=\frac{\pi}{4}\right)=\frac{1}{2}$, therefore for f(x) to be continuous at all the points in the interval $[0,\pi/2]$ should be assigned a value of $\frac{1}{2}$ at $x=\pi/4$.

Q56. Since the function $f(x)$ is continuous on the interval $[0,\infty)$ so, it's continuous at all the points belonging to this interval.

Continuity at x = 1 : $f(1)=a$ and, LHL (at x = 1) : $\lim\limits_{x\to1^-}\frac{x^2}{a}=\frac{1}{a}$

Since f is continuous at $x=1\in[0,\infty)$ so, $a=\frac{1}{a}$ $\Rightarrow a^2=1$ $\therefore a=\pm1$

Continuity at x = $\sqrt{2}$: $f(\sqrt{2})=\frac{2b^2-4b}{[\sqrt{2}]^2}=b^2-2b$ and, LHL (at x = $\sqrt{2}$) : $\lim\limits_{x\to\sqrt{2}^-} a=a$

Since f is continuous at $x=\sqrt{2}\in[0,\infty)$ so, $a=b^2-2b$

When $a=1: 1=b^2-2b$ $\Rightarrow b^2-2b-1=0$ $\therefore b=1\pm\sqrt{2}$

When $a=-1: -1=b^2-2b$ $\Rightarrow b^2-2b+1=0$ $\therefore b=1$.

Q57. As the function f is differentiable at x = 2, so it is continuous at x = 2 as well.

$\therefore \lim\limits_{x\to2^-}f(x)=\lim\limits_{x\to2^+}f(x)=f(2)$ $\Rightarrow \lim\limits_{x\to2^-}x^2=\lim\limits_{x\to2^+}ax+b=(2)^2 \Rightarrow 4=2a+b$ …(i)

Also, f is differentiable at x = 2 $\therefore Lf'(2)=Rf'(2)$ i.e., $\lim\limits_{x\to2^-}\frac{f(x)-f(2)}{x-2}=\lim\limits_{x\to2^+}\frac{f(x)-f(2)}{x-2}$

$\Rightarrow \lim\limits_{x\to2^-}\frac{x^2-4}{x-2}=\lim\limits_{x\to2^+}\frac{(ax+b)-4}{x-2}$ $\Rightarrow \lim\limits_{x\to2^-}(x+2)=\lim\limits_{x\to2^+}\frac{(ax+b)-4}{x-2}$ [by (i), $b=4-2a$

$\therefore 4=\lim\limits_{x\to2^+}\frac{(ax+4-2a)-4}{x-2}$ $\Rightarrow 4=\lim\limits_{x\to2^+}\frac{(x-2)a}{x-2}=\lim\limits_{x\to2^+}a$ $\Rightarrow a=4$

Replacing value of a in (i), we get : $b=-4$.

Q58. Since the continuity and differentiability of modulus function is doubtful at the corner points. So well shall check continuity and differentiability at the critical points x = 0, 1 $\in(-1,2)$.

$$\therefore f(x)=|x|+|x-1|=\begin{cases}-x-(x-1)=-2x+1, \text{ if } x<0\\ x-(x-1)=1, \text{ if } 0\le x<1\\ x+x-1=2x-1, \text{ if } x\ge 1\end{cases}$$

Continuity at x = 0 : We have f (0) = 1.

LHL (at x = 0) : $\lim_{x\to 0^-}(-2x+1)=-2\times 0+1=1$

RHL (at x = 0) : $\lim_{x\to 0^+}1=1$

Since $\lim_{x\to 0}f(x)=f(0)$ so f (x) is continuous at x = 0.

Continuity at x = 1 : We have f (1) = 2 (1) – 1 = 1.

LHL (at x = 1) : $\lim_{x\to 1^-}1=1$

RHL (at x = 1) : $\lim_{x\to 1^+}2x-1=2\times 1-1=1$

Since $\lim_{x\to 1}f(x)=f(1)$ so f (x) is continuous at x = 1.

Differentiability at x = 0 :

LHD (at x = 0) : $\lim_{x\to 0^-}\frac{|x|+|x-1|-1}{x-0}=\lim_{x\to 0^-}\frac{-2x+1-1}{x}=\lim_{x\to 0^-}\frac{-2x}{x}=-2$

RHD (at x = 0) : $\lim_{x\to 0^+}\frac{1-1}{x-0}=\lim_{x\to 0^+}\frac{0}{x}=\lim_{x\to 0^+}0=0\ne \text{LHD (at } x=0)$

$\therefore f(x)$ is not differentiable at $x=0$.

Differentiability at x = 1 :

LHD (at x = 1) : $\lim_{x\to 1^-}\frac{1-1}{x-1}=\lim_{x\to 1^-}\frac{0}{x-1}=\lim_{x\to 1^-}0=0$

RHD (at x = 1) : $\lim_{x\to 1^+}\frac{x+x-1-1}{x-1}=\lim_{x\to 1^+}\frac{2(x-1)}{x-1}=2\ne \text{LHD (at } x=1)$

$\therefore f(x)$ is not differentiable at $x=1$.

Alternative : Since the continuity and differentiability of modulus function is doubtful at the corner points. So well shall check continuity and differentiability at the critical points x = 0, 1 $\in(-1,2)$.

Continuity at x = 0 : We have f (0) = |0| + |0 – 1| = 1.

LHL (at x = 0) : $\lim_{x\to 0^-}|x|+|x-1|=|0|+|0-1|=1$

RHL (at x = 0) : $\lim_{x\to 0^+}|x|+|x-1|=|0|+|0-1|=1$

Since $\lim_{x\to 0}f(x)=f(0)$ so f (x) is continuous at x = 0.

Continuity at x = 1 : We have f (1) = |1| + |1 – 1| = 1.

LHL (at x = 1) : $\lim_{x\to 1^-}|x|+|x-1|=|1|+|1-1|=1$

RHL (at x = 1) : $\lim_{x\to 1^+}|x|+|x-1|=|1|+|1-1|=1$

Since $\lim_{x\to 1}f(x)=f(1)$ so f (x) is continuous at x = 1.

Differentiability at x = 0 :

LHD (at x = 0) : $\lim_{x\to 0^-}\frac{|x|+|x-1|-1}{x-0}=\lim_{x\to 0^-}\frac{-x-x+1-1}{x}=\lim_{x\to 0^-}\frac{-2x}{x}=-2$

RHD (at x = 0) : $\lim_{x\to 0^+}\frac{|x|+|x-1|-1}{x-0}=\lim_{x\to 0^+}\frac{x-x+1-1}{x}=\lim_{x\to 0^+}\frac{0}{x}=0\ne \text{LHD (at } x=0)$

$\therefore f(x)$ is not differentiable at $x=0$.

Differentiability at x = 1 :

LHD (at x = 1) : $\lim_{x\to1^-}\frac{|x|+|x-1|-1}{x-1}=\lim_{x\to1^-}\frac{x-x+1-1}{x-1}=\lim_{x\to1^-}\frac{0}{x-1}=0$

RHD (at x = 1) : $\lim_{x\to1^+}\frac{|x|+|x-1|-1}{x-1}=\lim_{x\to1^+}\frac{x+x-1-1}{x-1}=\lim_{x\to1^+}\frac{2(x-1)}{x-1}=2\neq \text{LHD (at } x=0)$

$\therefore f(x)$ is not differentiable at $x=1$.

Q59. We've $f(x)=x-|x-x^2|=\begin{cases} x-\left(-(x-x^2)\right)=2x-x^2, \text{if } -1\le x<0 \\ 0, \text{if } x=0 \\ x-(x-x^2)=x^2, \text{if } 0<x\le 1 \end{cases}$

Since the function f (x) is a polynomial function and is continuous on $[-1,0]\cup[0,1]$. That is, f (x) has one turning point (x = 0) in [–1, 1] so, we'll check its continuity at x = 0.

Continuity at x = 0 : We have $f(0)=0^2=0$

LHL (at $x=0$) $=\lim_{x\to0^-}f(x)=\lim_{x\to0^-}2x-x^2=2\times0-0^2=0$

RHL (at $x=0$) $=\lim_{x\to0^+}f(x)=\lim_{x\to0^+}x^2=0^2=0$

Since $\lim_{x\to0}f(x)=f(0)$, hence f (x) is continuous at x = 0.

Therefore, f (x) has no point of discontinuity on [–1, 1].

Q60. We have f (x) = |x – 3| + |x – 4| $=\begin{cases} -(x-3)-(x-4)=7-2x, \text{ if } x<3 \\ x-3-(x-4)=1, \text{ if } 3\le x<4 \\ x-3+x-4=2x-7, \text{ if } x\ge4 \end{cases}$

Differentiability at x = 3 :

LHD (at x = 3) : $\lim_{x\to3^-}\frac{7-2x-1}{x-3}=\lim_{x\to3^-}\frac{6-2x}{x-3}=\lim_{x\to0^-}\frac{-2(x-3)}{x-3}=-2$

RHD (at x = 3) : $\lim_{x\to3^+}\frac{1-1}{x-3}=\lim_{x\to3^+}\frac{0}{x-3}=0\neq Lf'(3)$

$\therefore f(x)$ is not differentiable at $x=3$.

Differentiability at x = 4 :

LHD (at x = 4) : $\lim_{x\to4^-}\frac{1-1}{x-4}=\lim_{x\to4^-}\frac{0}{x-4}=0$

RHD (at x = 4) : $\lim_{x\to4^+}\frac{2x-7-1}{x-4}=\lim_{x\to4^+}\frac{2(x-4)}{x-4}=2\neq \text{LHD (at } x=4)$

$\therefore f(x)$ is not differentiable at $x=4$.

Q61. We have $f(x)=\begin{cases} 5x-4 & , 0<x<1 \\ 4x^2-3x & , 1\le x<2 \\ 3x+4 & , x\ge2 \end{cases}$

Continuity at x = 1 :

LHL (at x = 1) : $\lim_{x\to1^-}f(x)=\lim_{x\to1^-}5x-4=5\times1-4=1$

RHL (at x = 1) : $\lim_{x\to1^+}f(x)=\lim_{x\to1^+}4x^2-3x=4\times1^2-3\times1=1$

Also, $f(1)=4(1)^3-3(1)=1$. Since $f(1)=\lim_{x\to1^-}f(x)=\lim_{x\to1^+}f(x)$ so, f is continuous at x = 1.

Differentiability at x = 2 :

We have $f(2)=3\times2+4=10$

LHD (at x = 2) : $\lim\limits_{x\to 2^-}\dfrac{f(x)-f(2)}{x-2}=\lim\limits_{x\to 2^-}\dfrac{4x^2-3x-10}{x-2}=\lim\limits_{x\to 2^-}\dfrac{(x-2)(4x+5)}{x-2}=\lim\limits_{x\to 2^-}(4x+5)=13$

RHD (at x = 2) : $\lim\limits_{x\to 2^+}\dfrac{3x+4-10}{x-2}=\lim\limits_{x\to 2^+}\dfrac{3(x-2)}{x-2}=\lim\limits_{x\to 2^+}3=3\neq \text{LHD (at } x=2)$

Hence $f(x)$ isn't differentiable at x = 2.

Note : This question had an error in the original question paper **(CBSE-2015 Guwahati)**. We have done the necessary corrections here before solving it.

Q62. We have Left Hand Limit at x = 0 : $\lim\limits_{x\to 0^-}\lambda(x^2+2)=\lambda(0^2+2)=2\lambda$

And, Right Hand Limit at x = 0 : $\lim\limits_{x\to 0^+}4x+6=4\times 0+6=6$

Since $f(x)$ is continuous at x = 0 so, LHL (at x = 0) = RHL (at x = 0) $\quad\therefore 2\lambda=6 \Rightarrow \lambda=3$.

Now, $f(0)=3(0^2+2)=6$

LHD (at x = 0) : $\lim\limits_{x\to 0^-}\dfrac{3(x^2+2)-6}{x-0}=\lim\limits_{x\to 0^-}\dfrac{3x^2+6-6}{x}=\lim\limits_{x\to 0^-}3x=0$

RHS (at x = 0) : $\lim\limits_{x\to 0^+}\dfrac{4x+6-6}{x-0}=\lim\limits_{x\to 0^-}\dfrac{4x}{x}=4\neq \text{LHD (at } x=0)$

Hence $f(x)$ isn't differentiable at x = 0.

Q63. We have $Lf'(1)=\lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^-}\dfrac{x-(2-1)}{x-1}=\lim\limits_{x\to 1^-}\dfrac{x-1}{x-1}=1,$

$Rf'(1)=\lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^+}\dfrac{2-x-(2-1)}{x-1}=\lim\limits_{x\to 1^+}\dfrac{-(x-1)}{x-1}=-1\neq Lf'(1)$

Hence $f(x)$ isn't differentiable at $x=1$.

$Lf'(2)=\lim\limits_{x\to 2^-}\dfrac{f(x)-f(2)}{x-2}=\lim\limits_{x\to 2^-}\dfrac{2-x-(2-2)}{x-2}=\lim\limits_{x\to 2^-}\dfrac{-(x-2)}{x-2}=-1,$

$Rf'(2)=\lim\limits_{x\to 2^+}\dfrac{f(x)-f(2)}{x-2}=\lim\limits_{x\to 2^+}\dfrac{-2+3x-x^2-(2-2)}{x-2}=\lim\limits_{x\to 2^+}\dfrac{-(x-2)(x-1)}{x-2}$

$\Rightarrow \quad =\lim\limits_{x\to 2^+}-(x-1)=-(2-1)=-1=Lf'(2)$

Hence $f(x)$ is differentiable at $x=2$.

Q64. Since $f(x)$ is continuous at $x=\pi/2$ so, $f(\pi/2)=\lim\limits_{x\to\pi/2}f(x)\ldots$(i)

Here $f(\pi/2)=p$...(ii)

LHL (at $x=\pi/2$) : $\lim\limits_{x\to\pi/2^-}f(x)=\lim\limits_{x\to\pi/2^-}\dfrac{1-\sin^3 x}{3\cos^2 x}=\lim\limits_{x\to\pi/2^-}\dfrac{(1-\sin x)(1+\sin x+\sin^2 x)}{3(1-\sin x)(1+\sin x)}$

$\Rightarrow \quad =\lim\limits_{x\to\pi/2^-}\dfrac{(1+\sin x+\sin^2 x)}{3(1+\sin x)}=\dfrac{1+\sin(\pi/2)+\sin^2(\pi/2)}{3(1+\sin(\pi/2))}=\dfrac{1}{2}\ldots$(iii) $\qquad [\because \sin(\pi/2)=1$

RHL (at $x=\pi/2$) : $\lim\limits_{x\to\pi/2^+}f(x)=\lim\limits_{x\to\pi/2^+}\dfrac{q(1-\sin x)}{(\pi-2x)^2}$ $\qquad$ Put $x=\dfrac{\pi}{2}+h$. As $x\to\dfrac{\pi}{2}\Rightarrow h\to 0$

$\Rightarrow \quad =\lim\limits_{h\to 0}\dfrac{q\left(1-\sin\left(\dfrac{\pi}{2}+h\right)\right)}{(-2h)^2}=\lim\limits_{h\to 0}\dfrac{q\left(1-\cos h\right)}{4h^2}=\lim\limits_{h\to 0}\dfrac{q\left(2\sin^2\dfrac{h}{2}\right)}{4\left(\dfrac{h^2}{4}\right)\times 4}=\dfrac{q}{8}$...(iv) $\quad [\because h\to 0\Rightarrow \dfrac{h}{2}\to 0$

By (i), (ii), (iii) & (iv), we get : $p=\dfrac{1}{2},\ \dfrac{q}{8}=\dfrac{1}{2} \qquad \therefore p=\dfrac{1}{2},\ q=4.$

Q65. Since $f(x)$ is continuous at $x=0$ so, $f(0)=\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^-}f(x)$

$\therefore f(0) = k\sin\frac{\pi}{2}(0+1) = k \qquad \Rightarrow k = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)...(i)$

Now RHL (at $x=0$) : $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\tan x - \sin x}{x^3}$

$$\Rightarrow \quad = \lim_{x\to 0^+} \frac{\sin x}{x} \times \frac{1-\cos x}{x^2} \times \frac{1}{\cos x}$$

$$\Rightarrow \quad = \lim_{x\to 0^+} \frac{\sin x}{x} \times 2\left(\frac{\sin^2(x/2)}{x^2/4}\right) \times \frac{1}{4} \times \frac{1}{\cos x} = 1\times 2(1)^2 \times \frac{1}{4} \times \frac{1}{\cos 0} = \frac{1}{2} \qquad [\because x\to 0 \ \therefore x/2 \to 0$$

By using (i), we get : $k = 1/2$.

Q66. LHL (at x = 0) : $\lim_{x\to 0^-}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ Put $x = 0-h$. As $x\to 0 \Rightarrow h\to 0$

$$\Rightarrow \quad = \lim_{h\to 0}\left(\frac{e^{-1/h}-1}{e^{-1/h}+1}\right) = \frac{e^{-1/0}-1}{e^{-1/0}+1} = \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{0-1}{0+1} = -1.$$

RHL (at x = 0) : $\lim_{x\to 0^+}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$ Put $x = 0+h$. As $x\to 0 \Rightarrow h\to 0$

$$\Rightarrow \quad = \lim_{h\to 0}\left(\frac{e^{1/h}-1}{e^{1/h}+1}\right) = \lim_{h\to 0}\left(\frac{1-e^{-1/h}}{1+e^{-1/h}}\right) = \frac{1-0}{1+0} = 1 \neq \text{ LHL (at } x = 0)$$

Therefore, $f(x)$ is discontinuous at $x = 0$.

Q67. LHL (at $x=0$) : $\lim_{x\to 0^-} \frac{\sin(a+1)x + 2\sin x}{x} = \lim_{x\to 0^-}\left(\frac{\sin(a+1)x}{(a+1)x}\right)\times(a+1) + 2\left(\frac{\sin x}{x}\right)$

$= 1\times(a+1) + 2\times 1 = a+3 \qquad$ [As $x\to 0 \ \therefore (a+1)x \to 0$

RHL (at $x=0$) : $\lim_{x\to 0^+} \frac{\sqrt{1+bx}-1}{x} = \lim_{x\to 0^+}\left(\frac{\sqrt{1+bx}-1}{x}\right)\times\frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1}$

$= \lim_{x\to 0^+}\left(\frac{1+bx-1}{x}\right)\times\frac{1}{\sqrt{1+bx}+1} = \lim_{x\to 0^+} b\times\frac{1}{\sqrt{1+bx}+1} = \frac{b}{2}$

Also, $f(0) = 2$

As $f(x)$ is continuous at $x = 0$ so, $a+3 = 2 = b/2 \qquad \therefore a = -1,\ b = 4$.