Chapter 02

SOLUTIONS Of EXERCISE FOR PRACTICE

Q01. **a)** $\cos^{-1}\frac{\sqrt{3}}{2}=\cos^{-1}\cos\frac{\pi}{6}=\frac{\pi}{6}$

b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)=-\sin^{-1}\frac{1}{\sqrt{2}}=-\sin^{-1}\sin\frac{\pi}{4}=-\frac{\pi}{4}$

c) $\tan^{-1}\frac{1}{\sqrt{3}}=\tan^{-1}\tan\frac{\pi}{6}=\frac{\pi}{6}$

d) $\text{cosec}^{-1}(2)=\text{cosec}^{-1}\left(\text{cosec}\frac{\pi}{6}\right)=\frac{\pi}{6}$

e) $\sec^{-1}\left(-\sqrt{2}\right)=\pi-\sec^{-1}\left(\sqrt{2}\right)=\pi-\sec^{-1}\left(\sec\frac{\pi}{4}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

f) $\cot^{-1}(1)=\cot^{-1}\cot(\pi/4)=\pi/4$

g) $\text{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)=\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)=-\frac{\pi}{3}$

h) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\cot^{-1}\left(\cot\frac{2\pi}{3}\right)=\frac{2\pi}{3}$

i) $\tan^{-1}\left(-\sqrt{3}\right)=-\tan^{-1}\left(\sqrt{3}\right)=-\tan^{-1}\left(\tan\frac{\pi}{3}\right)=-\frac{\pi}{3}$

j) $\sin^{-1}(0.5)=\sin^{-1}\frac{1}{2}=\sin^{-1}\sin\frac{\pi}{6}=\frac{\pi}{6}$

k) $\sec^{-1}(-1)=\sec^{-1}\sec\pi=\pi$

l) $\sin^{-1}(-1)=\sin^{-1}\sin\left(-\frac{\pi}{2}\right)=-\frac{\pi}{2}$

Q02. **a)** $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)=\tan^{-1}\left(\tan\frac{\pi}{3}\right)-\cot^{-1}\left(\cot\frac{5\pi}{6}\right)=\frac{\pi}{3}-\frac{5\pi}{6}=-\frac{\pi}{2}$

b) $\tan^{-1}(-1)+\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)+\cos^{-1}\left(\cos\frac{3\pi}{4}\right)=-\frac{\pi}{4}+\frac{3\pi}{4}=\frac{\pi}{2}$

c) $\tan^{-1}(1)+\sin^{-1}\left(-\frac{1}{2}\right)=\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\sin^{-1}\sin\left(-\frac{\pi}{6}\right)=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}$

d) $\sin^{-1}\frac{1}{2}-2\sin^{-1}\frac{1}{\sqrt{2}}=\sin^{-1}\sin\frac{\pi}{6}-2\sin^{-1}\sin\frac{\pi}{4}=\frac{\pi}{6}-2\times\frac{\pi}{4}=-\frac{\pi}{3}$

e) $2\sin^{-1}(-0.5)+\cos^{-1}(-0.5)=2\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$

$\Rightarrow\quad =\sin^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)+\frac{\pi}{2}=-\frac{\pi}{6}+\frac{\pi}{2}=\frac{\pi}{3}$

f) $\tan^{-1}(\sqrt{3})-\sec^{-1}(-2)=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}$

g) $\tan^{-1}(1)+\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{4}+\frac{\pi}{2}=\frac{3\pi}{4}$ $\quad\left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right.$

h) $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)+\operatorname{cosec}^{-1}\frac{2}{\sqrt{3}}=\frac{\pi}{3}-\frac{2\pi}{3}+\frac{\pi}{3}=0$

i) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}+\frac{\pi}{6}=-\frac{\pi}{6}$

j) $\tan^{-1}\left(-\sqrt{3}\right)-\cot^{-1}\left(\sqrt{3}\right)=-\frac{\pi}{3}-\frac{\pi}{6}=-\frac{\pi}{2}$

☞COMMIT TO MEMORY

#01. $1+\sin 2x=\cos^2 x+\sin^2 x+2\cos x\sin x=(\cos x+\sin x)^2$

#02. $1-\sin 2x=\cos^2 x+\sin^2 x-2\cos x\sin x=(\cos x-\sin x)^2$

#03. $\sqrt{1+\sin 2x}=\sqrt{\cos^2 x+\sin^2 x+2\cos x\sin x}=\sqrt{(\cos x+\sin x)^2}=|\cos x+\sin x|$

#04. $\sqrt{1-\sin 2x}=\sqrt{\cos^2 x+\sin^2 x-2\cos x\sin x}=\sqrt{(\cos x-\sin x)^2}=|\cos x-\sin x|$

#05. $\frac{\cos x+\sin x}{\cos x-\sin x}=\frac{1+\tan x}{1-\tan x}=\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}=\tan\left(\frac{\pi}{4}+x\right)$

#06. $\frac{\cos x-\sin x}{\cos x+\sin x}=\frac{1-\tan x}{1+\tan x}=\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}=\tan\left(\frac{\pi}{4}-x\right)$

Q03. **a)** RHS: Let $y=\sin^{-1}[3x-4x^3]$ Put $x=\sin\theta\Rightarrow\theta=\sin^{-1}x$...(i)

$\therefore\ y=\sin^{-1}[3\sin\theta-4\sin^3\theta]=\sin^{-1}\sin 3\theta=3\theta$

$\Rightarrow y=3\sin^{-1}x=\text{LHS}$ [by using (i)

b) Put $x=\cos\theta$ in RHS. Use $\cos 3\theta=4\cos^3\theta-3\cos\theta$.

c) LHS: Let $y=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ $\Rightarrow y=\tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right)$ $\Rightarrow y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=\text{RHS}$

d) LHS: Let $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ Put $x=\cos\theta\Rightarrow\theta=\cos^{-1}x$...(i)

$\therefore y=\sin^{-1}\left(2\cos\theta\sqrt{1-\cos^2\theta}\right)=\sin^{-1}\left(2\cos\theta\sin\theta\right)$

$\Rightarrow y=\sin^{-1}\left(\sin 2\theta\right)$ $\Rightarrow y=2\theta=2\cos^{-1}x=\text{RHS}$ [by using (i)

e) Put $x=\sin\theta$ in LHS

f) We have $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$ $\Rightarrow 2\tan^{-1}\sqrt{x}=\cos^{-1}\left(\frac{1-x}{1+x}\right)$

LHS: Let $y=2\tan^{-1}\sqrt{x}$ $\Rightarrow y=\cos^{-1}\left(\frac{1-[\sqrt{x}]^2}{1+[\sqrt{x}]^2}\right)$

$\therefore\ y=\cos^{-1}\left(\frac{1-x}{1+x}\right)=\text{RHS}$ $\qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right.$

g) LHS : Let $y=\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)=\tan^{-1}\left(\frac{\sqrt{2\cos^2\left(\frac{x}{2}\right)}+\sqrt{2\sin^2\left(\frac{x}{2}\right)}}{\sqrt{2\cos^2\left(\frac{x}{2}\right)}-\sqrt{2\sin^2\left(\frac{x}{2}\right)}}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{\sqrt{2}\,|\cos\frac{x}{2}|+\sqrt{2}\,|\sin\frac{x}{2}|}{\sqrt{2}\,|\cos\frac{x}{2}|-\sqrt{2}\,|\sin\frac{x}{2}|}\right)$ $\qquad \left[\because x\in\left(0,\frac{\pi}{2}\right)\Rightarrow 0<x<\frac{\pi}{2}\ \Rightarrow 0<\frac{x}{2}<\frac{\pi}{4}\right.$

$\Rightarrow y=\tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)=\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$

$\Rightarrow y=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)=\frac{\pi}{4}+\frac{x}{2}=\text{RHS}$

h) LHS : Let $y=\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)=\tan^{-1}\left(\frac{\sqrt{2\cos^2\left(\frac{x}{2}\right)}+\sqrt{2\sin^2\left(\frac{x}{2}\right)}}{\sqrt{2\cos^2\left(\frac{x}{2}\right)}-\sqrt{2\sin^2\left(\frac{x}{2}\right)}}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{|\cos\frac{x}{2}|+|\sin\frac{x}{2}|}{|\cos\frac{x}{2}|-|\sin\frac{x}{2}|}\right)$ $\qquad \left[\because \pi<x<\frac{3\pi}{2}\Rightarrow\frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}\right.$

$\therefore\ y=\tan^{-1}\left(\frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}}\right)=\tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$

$\Rightarrow y=\tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)=\frac{\pi}{4}-\frac{x}{2}=\text{RHS}$

i) LHS : Let $y=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$\Rightarrow y=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$

$\Rightarrow y=\cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1+\sin x}\sqrt{1-\sin x}}{(1+\sin x)-(1-\sin x)}\right)$ $\qquad \left[x\in\left(0,\frac{\pi}{2}\right)\therefore\sqrt{\cos^2 x}=|\cos x|=\cos x\right.$

$\Rightarrow y=\cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right)=\cot^{-1}\left(\frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}\right)$

$$\Rightarrow y=\cot^{-1}\left(\frac{\cos(x/2)}{\sin(x/2)}\right)=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=\text{RHS}.$$

Alternatively, $\text{LHS}:\cot^{-1}\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\cot^{-1}\dfrac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}$

$$\Rightarrow\ =\cot^{-1}\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\qquad\left[\because 0<x<\frac{\pi}{2}\ \therefore 0<\frac{x}{2}<\frac{\pi}{4}\right.$$

$$\Rightarrow\ =\cot^{-1}\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}=\cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)=\cot^{-1}\cot\frac{x}{2}=\frac{x}{2}=\text{RHS}.$$

j) LHS : Let $y=\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\times\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

$$\Rightarrow y=\tan^{-1}\left(\frac{1+x+1-x-2\sqrt{1+x}\sqrt{1-x}}{(1+x)-(1-x)}\right)=\tan^{-1}\left(\frac{2[1-\sqrt{1-x^2}]}{2x}\right)=\tan^{-1}\left(\frac{1-\sqrt{1-x^2}}{x}\right)$$

Put $x=\sin\theta\Rightarrow\theta=\sin^{-1}x$...(i) $\qquad\therefore y=\tan^{-1}\left(\dfrac{1-\sqrt{1-\sin^2\theta}}{\sin\theta}\right)=\tan^{-1}\left(\dfrac{1-|\cos\theta|}{\sin\theta}\right)$

$$\left[\because -\frac{1}{\sqrt2}\le x\le 1\Rightarrow -\frac{1}{\sqrt2}\le\sin\theta\le 1\Rightarrow \sin^{-1}\left(-\frac{1}{\sqrt2}\right)\le\sin^{-1}\sin\theta\le\sin^{-1}1\Rightarrow -\frac{\pi}{4}\le\theta\le\frac{\pi}{2}\right]$$

$$\therefore y=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\tan^{-1}\left(\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)=\tan^{-1}\left(\frac{\sin(\theta/2)}{\cos(\theta/2)}\right)=\tan^{-1}\tan\frac{\theta}{2}=\frac{\theta}{2}$$

$\Rightarrow y=\dfrac{1}{2}\sin^{-1}x\qquad$ [by using (i)

$$\therefore y=\frac{1}{2}\left[\frac{\pi}{2}-\cos^{-1}x\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x=\text{RHS}$$

k) LHS : Let $y=\tan^{-1}\left(\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

Put $x=\cos\theta\Rightarrow\theta=\cos^{-1}x$...(i) $\qquad\therefore y=\tan^{-1}\left(\dfrac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$

$$\Rightarrow y=\tan^{-1}\left(\frac{\sqrt2\left|\cos\frac{\theta}{2}\right|+\sqrt2\left|\sin\frac{\theta}{2}\right|}{\sqrt2\left|\cos\frac{\theta}{2}\right|-\sqrt2\left|\sin\frac{\theta}{2}\right|}\right)\qquad\Rightarrow y=\tan^{-1}\left(\frac{\sqrt2\left|\cos\frac{\theta}{2}\right|+\sqrt2\left|\sin\frac{\theta}{2}\right|}{\sqrt2\left|\cos\frac{\theta}{2}\right|-\sqrt2\left|\sin\frac{\theta}{2}\right|}\right)$$

$$\left[\because -\frac{1}{\sqrt2}\le x\le 1\Rightarrow -\frac{1}{\sqrt2}\le\cos\theta\le 1\Rightarrow \cos^{-1}(1)\le\cos^{-1}\cos\theta\le\cos^{-1}\left(-\frac{1}{\sqrt2}\right)\right.$$
$$\left.\Rightarrow 0\le\theta\le\frac{3\pi}{4}\Rightarrow 0\le\frac{\theta}{2}\le\frac{3\pi}{8}\qquad\therefore\frac{\theta}{2}\ \text{lies in I Quadrant}\right]$$

So, $y = \tan^{-1}\left(\dfrac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right) = \tan^{-1}\left(\dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right) = \tan^{-1}\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$

$\Rightarrow y = \dfrac{\pi}{4}+\dfrac{\theta}{2} = \dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x = \text{RHS}$

l) LHS : Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ Put $x^2 = \cos 2\theta \Rightarrow \theta = \dfrac{1}{2}\cos^{-1}x^2 \ldots$(i)

$\therefore y = \tan^{-1}\left(\dfrac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right) = \tan^{-1}\left(\dfrac{|\cos\theta|+|\sin\theta|}{|\cos\theta|-|\sin\theta|}\right)$

$$\left[\begin{array}{c} \because -1 \le x \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow 0 \le \cos 2\theta \le 1 \\ \Rightarrow \cos^{-1}1 \le \cos^{-1}\cos 2\theta \le \cos^{-1}0 \Rightarrow 0 \le 2\theta \le \dfrac{\pi}{2} \Rightarrow 0 \le \theta \le \dfrac{\pi}{4} \end{array}\right]$$

$\Rightarrow y = \tan^{-1}\left(\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left(\tan\left(\dfrac{\pi}{4}+\theta\right)\right)$

$\therefore y = \dfrac{\pi}{4}+\theta = \dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2 = \text{RHS}$ [by using (i)]

m) Let $y = \sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]$

Put $\tan^{-1}x = \alpha \Rightarrow \tan\alpha = x \Rightarrow \cos\alpha = \dfrac{1}{\sqrt{x^2+1}} \Rightarrow \alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{x^2+1}}\right)$

$\therefore \cos(\tan^{-1}x) = \cos\left(\cos^{-1}\dfrac{1}{\sqrt{x^2+1}}\right) = \dfrac{1}{\sqrt{x^2+1}}$ $\Rightarrow y = \sin\left[\cot^{-1}\dfrac{1}{\sqrt{x^2+1}}\right]$

Now put $\cot^{-1}\dfrac{1}{\sqrt{x^2+1}} = \beta \Rightarrow \cot\beta = \dfrac{1}{\sqrt{x^2+1}} \Rightarrow \sin\beta = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$ $\Rightarrow \beta = \sin^{-1}\sqrt{\dfrac{x^2+1}{x^2+2}}$

So, $y = \sin\left[\cot^{-1}\left(\dfrac{1}{\sqrt{x^2+1}}\right)\right] = \sin\left[\sin^{-1}\left(\sqrt{\dfrac{x^2+1}{x^2+2}}\right)\right] = \sqrt{\dfrac{x^2+1}{x^2+2}}$

$\therefore \sin[\cot^{-1}\{\cos(\tan^{-1}x)\}] = \sqrt{\dfrac{x^2+1}{x^2+2}}$

n) Similar as last question. Put $\cot^{-1}x = \theta$ and proceed.

o) Let $x = \sin\theta, y = \sin\beta \Rightarrow \theta = \sin^{-1}x, \beta = \sin^{-1}y$ in RHS. …(i)

$\therefore \sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} = \sin^{-1}\left\{\sin\theta\sqrt{1-\sin^2\beta}+\sin\beta\sqrt{1-\sin^2\theta}\right\}$

$\Rightarrow \quad = \sin^{-1}\{\sin\theta\cos\beta+\sin\beta\cos\theta\} = \sin^{-1}\sin(\theta+\beta)$

$\Rightarrow \quad = \theta+\beta = \sin^{-1}x+\sin^{-1}y = \text{LHS}$ [by using (i)]

NOTE : This relation can be used as an identity in certain questions.

Also, prove yourself : $\sin^{-1}x-\sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$.

p) Let $x = \cos\theta, y = \cos\beta \Rightarrow \theta = \cos^{-1}x, \beta = \cos^{-1}y$ in RHS. …(i)

$\therefore \cos^{-1}\left\{xy-\sqrt{(1-x^2)(1-y^2)}\right\} = \cos^{-1}\left\{\cos\theta\cos\beta-\sqrt{1-\cos^2\theta}\sqrt{1-\cos^2\beta}\right\}$

$\Rightarrow \quad = \cos^{-1}\{\cos\theta\cos\beta - \sin\theta\sin\beta\}$ $\quad \Rightarrow \quad = \cos^{-1}\{\cos(\theta+\beta)\}$

$\Rightarrow \quad \theta+\beta = \cos^{-1}x + \cos^{-1}y = \text{LHS}$ $\quad$ [by using (i)

NOTE : This relation can be used as an identity in certain questions.

Also, prove yourself : $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left\{xy + \sqrt{(1-x^2)(1-y^2)}\right\}$.

q) LHS : Let $y = \sin^{-1}\left[\dfrac{\sqrt{1+x}+\sqrt{1-x}}{2}\right]$ $\quad$ Put $x = \sin 2\theta \Rightarrow \theta = \dfrac{1}{2}\sin^{-1}x$...(i)

$\therefore y = \sin^{-1}\left[\dfrac{\sqrt{1+\sin 2\theta}+\sqrt{1-\sin 2\theta}}{2}\right]$

$\Rightarrow y = \sin^{-1}\left[\dfrac{\sqrt{\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}+\sqrt{\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta}}{2}\right]$

$\Rightarrow y = \sin^{-1}\left[\dfrac{\sqrt{(\cos\theta+\sin\theta)^2}+\sqrt{(\cos\theta-\sin\theta)^2}}{2}\right] = \sin^{-1}\left[\dfrac{|\cos\theta+\sin\theta|+|\cos\theta-\sin\theta|}{2}\right]$

$\left[\because 0 < x < 1 \Rightarrow 0 < \sin 2\theta < 1 \Rightarrow 0 < \theta < \dfrac{\pi}{4}\right]$

$\therefore y = \sin^{-1}\left[\dfrac{\cos\theta+\sin\theta+\cos\theta-\sin\theta}{2}\right] = \sin^{-1}\cos\theta = \sin^{-1}\sin\left[\dfrac{\pi}{2}-\theta\right]$

$\Rightarrow y = \dfrac{\pi}{2} - \dfrac{1}{2}\sin^{-1}x = \text{RHS}$ $\quad$ [by using (i)

r) LHS : Let $y = \sin^{-1}\left[\dfrac{\sin x+\cos x}{\sqrt{2}}\right] \Rightarrow y = \sin^{-1}\left[\dfrac{1}{\sqrt{2}}\sin x + \dfrac{1}{\sqrt{2}}\cos x\right]$

$\Rightarrow y = \sin^{-1}\left[\cos\dfrac{\pi}{4}\sin x + \sin\dfrac{\pi}{4}\cos x\right] = \sin^{-1}\left[\sin\left(x+\dfrac{\pi}{4}\right)\right] = x + \dfrac{\pi}{4} = \text{RHS}$

s) LHS : Let $y = \tan^{-1}\left(\dfrac{m}{n}\right) - \tan^{-1}\left(\dfrac{m-n}{m+n}\right)$ $\quad \Rightarrow y = \tan^{-1}\left(\dfrac{m}{n}\right) - \tan^{-1}\left(\dfrac{1-\dfrac{n}{m}}{1+\dfrac{n}{m}}\right)$

$\Rightarrow y = \tan^{-1}\left(\dfrac{m}{n}\right) - \left(\tan^{-1}1 - \tan^{-1}\dfrac{n}{m}\right) = \tan^{-1}\dfrac{m}{n} - \tan^{-1}1 + \tan^{-1}\dfrac{n}{m}$

$\Rightarrow y = \dfrac{\pi}{2} - \cot^{-1}\dfrac{m}{n} - \dfrac{\pi}{4} + \cot^{-1}\dfrac{m}{n}$ $\quad \therefore y = \dfrac{\pi}{4} = \text{RHS}$

t) Consider LHS and put $\theta = \dfrac{1}{2}\cos^{-1}\dfrac{a}{b} \Rightarrow \cos 2\theta = \dfrac{a}{b}$...(i)

LHS : Let $y = \tan\left[\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right] + \tan\left[\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right] = \tan\left[\dfrac{\pi}{4}+\theta\right] + \tan\left[\dfrac{\pi}{4}-\theta\right]$

$\Rightarrow y = \dfrac{\tan\dfrac{\pi}{4}+\tan\theta}{1-\tan\dfrac{\pi}{4}\tan\theta} + \dfrac{\tan\dfrac{\pi}{4}-\tan\theta}{1+\tan\dfrac{\pi}{4}\tan\theta}$ $\quad \Rightarrow y = \dfrac{1+\tan\theta}{1-\tan\theta} + \dfrac{1-\tan\theta}{1+\tan\theta}$

$\Rightarrow y = \dfrac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{1-\tan^2\theta}$ $\qquad \Rightarrow y = \dfrac{2(1+\tan^2\theta)}{1-\tan^2\theta} = 2\left(\dfrac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$\Rightarrow y = 2\left(\dfrac{1}{\cos 2\theta}\right) = 2\left(\dfrac{1}{a/b}\right)$ $\qquad \therefore y = \dfrac{2b}{a} = \text{RHS}$ $\qquad$ [by using (i)

Try yourself : $\tan\left[\dfrac{\pi}{4} + \dfrac{1}{2}\tan^{-1}\dfrac{a}{b}\right] + \tan\left[\dfrac{\pi}{4} - \dfrac{1}{2}\tan^{-1}\dfrac{a}{b}\right] = \dfrac{2\sqrt{a^2+b^2}}{b}$.

u) Consider LHS and put $\cos^{-1}x = \theta$ $\qquad \Rightarrow x = \cos\theta$

Let $y = \sin\cot^{-1}\tan\cos^{-1}x = \sin\cot^{-1}\tan\theta$ $\quad \Rightarrow y = \sin\cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

Put $\cot^{-1}\dfrac{\sqrt{1-x^2}}{x} = \alpha$ $\qquad \Rightarrow \cot\alpha = \dfrac{\sqrt{1-x^2}}{x}$ $\qquad \Rightarrow \sin\alpha = \dfrac{x}{1} = y$.

$\therefore\ y = \sin\cot^{-1}\tan\cos^{-1}x = x = \text{RHS}$.

v) LHS : Let $y = \tan^{-1}\left(\dfrac{\cos x}{1-\sin x}\right) - \cot^{-1}\left(\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}\right) - \cot^{-1}\left(\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}\right) - \cot^{-1}\left(\left|\cot\frac{x}{2}\right|\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right) - \cot^{-1}\left(\cot\frac{x}{2}\right) \qquad \left[\because 0 < x < \frac{\pi}{2} \Rightarrow 0 < x < \frac{\pi}{4}\right]$$

$$\Rightarrow y = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) - \frac{x}{2} = \tan^{-1}\tan\left[\frac{\pi}{4} + \frac{x}{2}\right] - \frac{x}{2}$$

$\therefore y = \dfrac{\pi}{4} + \dfrac{x}{2} - \dfrac{x}{2} = \dfrac{\pi}{4} = \text{RHS}$

w) Consider $y = \tan^{-1}\left(\dfrac{x}{\sqrt{a^2-x^2}}\right)$ $\qquad$ Put $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\dfrac{x}{a}\right)$ …(i)

$\therefore y = \tan^{-1}\left(\dfrac{a\sin\theta}{\sqrt{a^2(1-\sin^2\theta)}}\right) = \tan^{-1}\left(\dfrac{a\sin\theta}{a\cos\theta}\right) = \theta = \sin^{-1}\dfrac{x}{a}$, Proved $\qquad$ [by using (i)

Also $y = \cos^{-1}\left(\dfrac{\sqrt{a^2-x^2}}{a}\right) = \cos^{-1}\left(\dfrac{\sqrt{a^2(1-\sin^2\theta)}}{a}\right) = \cos^{-1}\left(\dfrac{a\cos\theta}{a}\right) = \theta = \sin^{-1}\dfrac{x}{a}$, Proved

Again $y = \cot^{-1}\left(\dfrac{\sqrt{a^2(1-\sin^2\theta)}}{a\sin\theta}\right) = \cot^{-1}\left(\dfrac{a\cos\theta}{a\sin\theta}\right) = \theta = \sin^{-1}\dfrac{x}{a}$, Proved

$\therefore\ \tan^{-1}\left(\dfrac{x}{\sqrt{a^2-x^2}}\right)=\sin^{-1}\left(\dfrac{x}{a}\right)=\cos^{-1}\left(\dfrac{\sqrt{a^2-x^2}}{a}\right)=\cot^{-1}\left(\dfrac{\sqrt{a^2-x^2}}{x}\right)$ Hence Proved.

x) Consider $y=2\sin^{-1}\sqrt{\dfrac{1-x}{2}}$ Put $x=\cos\theta\Rightarrow\theta=\cos^{-1}x\ \ldots(i)$

$\Rightarrow y=2\sin^{-1}\sqrt{\dfrac{1-\cos\theta}{2}}=2\sin^{-1}\sin\dfrac{\theta}{2}=\theta=\cos^{-1}x$, Proved

Also $y=2\cos^{-1}\sqrt{\dfrac{1+x}{2}}=2\cos^{-1}\sqrt{\dfrac{1+\cos\theta}{2}}=2\cos^{-1}\cos\dfrac{\theta}{2}=\theta=\cos^{-1}x$, Proved

$\therefore\cos^{-1}x=2\sin^{-1}\sqrt{\dfrac{1-x}{2}}=2\cos^{-1}\sqrt{\dfrac{1+x}{2}}$ Hence proved.

y) LHS : Let $y=\cot\left(\tan^{-1}x+\tan^{-1}\dfrac{1}{x}\right)+\cos^{-1}\left(1-2x^2\right)+\cos^{-1}\left(2x^2-1\right)$

$\Rightarrow\quad=\cot\left(\tan^{-1}x+\cot^{-1}x\right)+\cos^{-1}\left(1-2x^2\right)+\cos^{-1}\left(-(1-2x^2)\right)$

$\Rightarrow\quad=\cot\left(\pi/2\right)+\cos^{-1}\left(1-2x^2\right)+\pi-\cos^{-1}\left(1-2x^2\right)=\pi=\text{RHS}$

z) LHS : Let $y=\tan^{-1}\left(\dfrac{a-b}{1+ab}\right)+\tan^{-1}\left(\dfrac{b-c}{1+bc}\right)+\tan^{-1}\left(\dfrac{c-a}{1+ca}\right)$

$\Rightarrow\quad=\tan^{-1}a-\tan^{-1}b+\tan^{-1}b-\tan^{-1}c+\tan^{-1}c-\tan^{-1}a=0=\text{RHS}$

aa) LHS : $\cot^{-1}\left(\dfrac{xy+1}{x-y}\right)+\cot^{-1}\left(\dfrac{yz+1}{y-z}\right)+\cot^{-1}\left(\dfrac{zx+1}{z-x}\right)$

$\Rightarrow\quad=\tan^{-1}\left(\dfrac{x-y}{1+xy}\right)+\tan^{-1}\left(\dfrac{y-z}{1+yz}\right)+\tan^{-1}\left(\dfrac{z-x}{1+zx}\right)$

$\Rightarrow\quad=\tan^{-1}x-\tan^{-1}y+\tan^{-1}y-\tan^{-1}z+\tan^{-1}z-\tan^{-1}x$

$\Rightarrow\quad=0=\text{RHS}$

ab) LHS : $\sin\left[\tan^{-1}\left(\dfrac{1-x^2}{2x}\right)+\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)\right]=\sin\left[\cot^{-1}\left(\dfrac{2x}{1-x^2}\right)+2\tan^{-1}x\right]$

$\Rightarrow\quad=\sin\left[\dfrac{\pi}{2}-\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)+2\tan^{-1}x\right]=\sin\left[\dfrac{\pi}{2}-2\tan^{-1}x+2\tan^{-1}x\right]$

$\Rightarrow\quad=\sin\dfrac{\pi}{2}=1=\text{RHS.}$

ac) LHS : Let $y=\tan^{-1}\left(\dfrac{6x-8x^3}{1-12x^2}\right)-\tan^{-1}\left(\dfrac{4x}{1-4x^2}\right)$ Put $2x=\tan\theta\Rightarrow\theta=\tan^{-1}2x\ldots(i)$

$\therefore y=\tan^{-1}\left(\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)-\tan^{-1}\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)$ $\Rightarrow y=\tan^{-1}\left(\tan 3\theta\right)-\tan^{-1}\left(\tan 2\theta\right)$

$\Rightarrow y=3\theta-2\theta=\theta=\tan^{-1}2x=\text{RHS}$. [By (i)

Q04. **a)** Let $y=\cot^{-1}\left[\dfrac{1}{\sqrt{x^2-1}}\right]$ Put $x=\sec\theta\Rightarrow\theta=\sec^{-1}x\ldots(i)$

$\Rightarrow y=\cot^{-1}\left[\dfrac{1}{\sqrt{\sec^2\theta-1}}\right]=\cot^{-1}\left[\dfrac{1}{\tan\theta}\right]=\cot^{-1}\cot\theta=\theta$

$\therefore y=\sec^{-1}x$. [by using (i)

Another similar sum : Simplify $\cot^{-1}\dfrac{1}{\sqrt{x^2-1}}$ for $x<-1$.

Sol. Let $y=\cot^{-1}\dfrac{1}{\sqrt{x^2-1}},\ x<-1$.

Put $x=\sec\theta \Rightarrow \theta=\sec^{-1}x$. Then $\sec\theta<-1 \Rightarrow \dfrac{\pi}{2}<\theta<\pi$

$\therefore\ y=\cot^{-1}\dfrac{1}{\sqrt{\sec^2\theta-1}}=\cot^{-1}\left(\dfrac{1}{|\tan\theta|}\right) \qquad \Rightarrow y=\cot^{-1}\left(\dfrac{1}{-\tan\theta}\right)=\cot^{-1}(-\cot\theta)$

$\Rightarrow y=\cot^{-1}(\cot(\pi-\theta))=\pi-\theta \qquad \therefore y=\pi-\sec^{-1}x$.

b) Let $y=\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right) \qquad \Rightarrow y=\tan^{-1}\left(\sqrt{\dfrac{2\sin^2(x/2)}{2\cos^2(x/2)}}\right)$

$\Rightarrow y=\tan^{-1}\left|\tan\dfrac{x}{2}\right| \qquad \left[\because -\pi<x<\pi \Rightarrow -\dfrac{\pi}{2}<\dfrac{x}{2}<\dfrac{\pi}{2}\right]$

$$\therefore y=\tan^{-1}\left|\tan\frac{x}{2}\right|=\begin{cases}\tan^{-1}\left(-\tan\dfrac{x}{2}\right), & \text{if } -\pi<x<0\\ \tan^{-1}\left(\tan\dfrac{x}{2}\right), & \text{if } 0\le x<\pi\end{cases}=\begin{cases}\tan^{-1}\left[\tan\left(-\dfrac{x}{2}\right)\right]=-\dfrac{x}{2}, & \text{if } -\pi<x<0\\ \tan^{-1}\left[\tan\left(\dfrac{x}{2}\right)\right]=\dfrac{x}{2}, & \text{if } 0\le x<\pi\end{cases}$$

c) Let $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right) \qquad$ Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$...(i)

$\Rightarrow y=\tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)=\tan^{-1}\left(\dfrac{\sec\theta-1}{\tan\theta}\right)$

$\Rightarrow y=\tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)=\tan^{-1}\dfrac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\tan^{-1}\tan\dfrac{\theta}{2}$

$\therefore y=\dfrac{\theta}{2}=\dfrac{1}{2}\tan^{-1}x \qquad$ [by using (i)

d) Let $y=\cos^{-1}\left(\dfrac{3}{5}\cos x+\dfrac{4}{5}\sin x\right) \qquad$ Put $\cos\theta=\dfrac{3}{5},\sin\theta=\dfrac{4}{5}\Rightarrow\theta=\tan^{-1}\dfrac{4}{3}$ …(i)

$\Rightarrow y=\cos^{-1}(\cos\theta\cos x+\sin\theta\sin x) \qquad \Rightarrow y=\cos^{-1}\cos(\theta-x)$

$\therefore y=\theta-x=\tan^{-1}\dfrac{4}{3}-x$.

Also, simplify yourself : $\sin^{-1}\left(\dfrac{3\sin x+2\cos x}{\sqrt{13}}\right)$. Answer : $x+\tan^{-1}\dfrac{2}{3}$.

e) Let $y=\tan^{-1}\left(\dfrac{x}{\sqrt{a^2-x^2}}\right) \qquad$ Put $x=a\sin\theta\Rightarrow\theta=\sin^{-1}\dfrac{x}{a}$ …(i)

$\Rightarrow y=\tan^{-1}\left(\dfrac{a\sin\theta}{\sqrt{a^2(1-\sin^2\theta)}}\right)=\tan^{-1}\left(\dfrac{a\sin\theta}{a\cos\theta}\right)=\tan^{-1}\tan\theta$

$\therefore y = \theta = \sin^{-1}\frac{x}{a}$ [by using (i)

f) Let $y = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$ $\Rightarrow y = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right) = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right) \qquad \Rightarrow y = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) = \tan^{-1}\tan\left[\frac{\pi}{4}+\frac{x}{2}\right] = \frac{\pi}{4}+\frac{x}{2}$$

g) Let $y = \tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right) = \tan^{-1}\left(\frac{\cos x\left(1-\frac{\sin x}{\cos x}\right)}{\cos x\left(1+\frac{\sin x}{\cos x}\right)}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left(\frac{\tan(\pi/4)-\tan x}{1+\tan(\pi/4)\tan x}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}-x\right) = \frac{\pi}{4}-x$$

h) Let $y = \sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$ $\Rightarrow y = \sin^{-1}\left\{x\sqrt{1-(\sqrt{x})^2}-\sqrt{x}\sqrt{1-x^2}\right\}$

$$\Rightarrow y = \sin^{-1}\left\{x\sqrt{1-(\sqrt{x})^2}-\sqrt{x}\sqrt{1-x^2}\right\}$$

Put $x = \sin\theta, \sqrt{x} = \sin\beta \Rightarrow \theta = \sin^{-1}x, \beta = \sin^{-1}\sqrt{x}$ ...(i)

$$\therefore y = \sin^{-1}\left\{\sin\theta\sqrt{1-\sin^2\beta}-\sin\beta\sqrt{1-\sin^2\theta}\right\}$$

$$\Rightarrow y = \sin^{-1}\{\sin\theta\cos\beta-\sin\beta\cos\theta\} = \sin^{-1}\{\sin(\theta-\beta)\} = \theta-\beta$$

$$\therefore y = \sin^{-1}x-\sin^{-1}\sqrt{x}$$

i) Let $y = \tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$

Divide Nr & Dr by $b\cos x$ $\Rightarrow y = \tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right) = \tan^{-1}\frac{a}{b}-\tan^{-1}\tan x$

$\therefore y = \tan^{-1}\frac{a}{b}-x$. [By comparing to $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x-\tan^{-1}y$.

j) Let $y = \tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$ $\Rightarrow y = \tan^{-1}\left(\frac{3\frac{x}{a}-\frac{x^3}{a^3}}{1-3\frac{x^2}{a^2}}\right)$

Put $\frac{x}{a} = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$...(i) $\Rightarrow y = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$\Rightarrow y = \tan^{-1}\tan 3\theta = 3\theta$ $\therefore y = 3\tan^{-1}\frac{x}{a}$ [by using (i)

k) Let $y=\tan^{-1}[x+\sqrt{1+x^2}]$ Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$...(i)

$\Rightarrow y=\tan^{-1}[\tan\theta+\sqrt{1+\tan^2\theta}]$ $\Rightarrow y=\tan^{-1}[\tan\theta+\sec\theta]$

$\Rightarrow y=\tan^{-1}\left[\frac{1+\sin\theta}{\cos\theta}\right]$ $\Rightarrow y=\tan^{-1}\left[\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}\right]$

$$\Rightarrow y=\tan^{-1}\left[\frac{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}\right]=\tan^{-1}\left[\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right]$$

$$\Rightarrow y=\tan^{-1}\left[\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right]=\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{\pi}{4}+\frac{\theta}{2}$$

$\therefore y=\frac{\pi}{4}+\frac{1}{2}\tan^{-1}x$ [by using (i)

l) Let $y=\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$ Put $x=\cos\theta$ ...(i)

$\Rightarrow y=\sin\left[2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]$ $\Rightarrow y=\sin\left[2\tan^{-1}\sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\right]$

$\Rightarrow y=\sin\left[2\tan^{-1}\tan\frac{\theta}{2}\right]=\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-x^2}$ [by using (i)

m) Let $y=\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}$ Put $x=\sin\theta \Rightarrow \theta=\sin^{-1}x$...(i)

$$\Rightarrow y=\sin^{-1}\left\{\frac{\sin\theta+\sqrt{1-\sin^2\theta}}{\sqrt{2}}\right\}=\sin^{-1}\left\{\frac{\sin\theta+|\cos\theta|}{\sqrt{2}}\right\}$$

$\Rightarrow y=\sin^{-1}\left\{\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right\}$ $\left[\because -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{4}<\theta<\frac{\pi}{4}\right.$

$$\Rightarrow y=\sin^{-1}\left\{\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right\}=\sin^{-1}\sin\left(\theta+\frac{\pi}{4}\right)$$

$\therefore y=\sin^{-1}x+\frac{\pi}{4}$ [by using (i)

n) Let $y=\tan^{-1}\left(\sqrt{\frac{a-x}{a+x}}\right)$ Put $x=a\cos\theta \Rightarrow \theta=\cos^{-1}\frac{x}{a}$ ...(i)

$$\Rightarrow y=\tan^{-1}\left(\sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right)$$

$\Rightarrow y=\tan^{-1}\tan\frac{\theta}{2}=\frac{\theta}{2}$ $\therefore y=\frac{1}{2}\cos^{-1}\frac{x}{a}$ [by using (i)

o) Let $y=\sin^{-1}\left[\dfrac{x}{\sqrt{x^2+a^2}}\right]$ Put $x=a\tan\theta\Rightarrow\theta=\tan^{-1}\dfrac{x}{a}\ldots$(i)

$\Rightarrow y=\sin^{-1}\left[\dfrac{a\tan\theta}{\sqrt{a^2\tan^2\theta+a^2}}\right]$ $\Rightarrow y=\sin^{-1}\left[\dfrac{a\tan\theta}{a\sec\theta}\right]$

$\Rightarrow y=\sin^{-1}\sin\theta=\theta$ $\therefore y=\tan^{-1}(x/a)$ [by using (i)

p) Let $y=\cot^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}-\tan^{-1}\left[\dfrac{1-\sin x}{\cos x}\right]$

$\Rightarrow y=\cot^{-1}\sqrt{\dfrac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}-\tan^{-1}\left[\dfrac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}{\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}\right]=\cot^{-1}\tan\dfrac{x}{2}-\tan^{-1}\left[\dfrac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right]$

$\Rightarrow y=\cot^{-1}\cot\left(\dfrac{\pi}{2}-\dfrac{x}{2}\right)-\tan^{-1}\left[\dfrac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right]=\dfrac{\pi}{2}-\dfrac{x}{2}-\tan^{-1}\tan\left[\dfrac{\pi}{4}-\dfrac{x}{2}\right]$

$\Rightarrow y=\dfrac{\pi}{2}-\dfrac{x}{2}-\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)$ $\therefore y=\dfrac{\pi}{4}$.

q) Let $y=\cos^{-1}\sqrt{\dfrac{\sqrt{1+x^2}+1}{2\sqrt{1+x^2}}}$ Put $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$...(i)

$\Rightarrow y=\cos^{-1}\sqrt{\dfrac{\sqrt{1+\tan^2\theta}+1}{2\sqrt{1+\tan^2\theta}}}=\cos^{-1}\sqrt{\dfrac{\sec\theta+1}{2\sec\theta}}$ $\Rightarrow y=\cos^{-1}\sqrt{\dfrac{1+\cos\theta}{2}}=\cos^{-1}\sqrt{\dfrac{2\cos^2(\theta/2)}{2}}$

$\Rightarrow y=\cos^{-1}\cos(\theta/2)=\dfrac{1}{2}\theta$ $\therefore y=\dfrac{1}{2}\tan^{-1}x$ [by using (i)

r) Let $y=\tan^{-1}\left(\dfrac{3\sin 2x}{5+3\cos 2x}\right)+\tan^{-1}\left(\dfrac{1}{4}\tan x\right)$

$\Rightarrow \quad =\tan^{-1}\left(\dfrac{3\left(\dfrac{2\tan x}{1+\tan^2 x}\right)}{5+3\left(\dfrac{1-\tan^2 x}{1+\tan^2 x}\right)}\right)+\tan^{-1}\left(\dfrac{1}{4}\tan x\right)$

$\Rightarrow \quad =\tan^{-1}\left(\dfrac{6\tan x}{5+5\tan^2 x+3-3\tan^2 x}\right)+\tan^{-1}\left(\dfrac{1}{4}\tan x\right)$

$\Rightarrow \quad =\tan^{-1}\left(\dfrac{3\tan x}{4+\tan^2 x}\right)+\tan^{-1}\left(\dfrac{1}{4}\tan x\right)$ $\Rightarrow \quad =\tan^{-1}\left(\dfrac{\dfrac{3\tan x}{4+\tan^2 x}+\dfrac{1}{4}\tan x}{1-\dfrac{3\tan x}{4+\tan^2 x}\times\dfrac{1}{4}\tan x}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\dfrac{16\tan x+\tan^3 x}{16+4\tan^2 x-3\tan^2 x}\right)$ $\Rightarrow \quad =\tan^{-1}\left(\dfrac{\tan x(16+\tan^2 x)}{16+\tan^2 x}\right)$

$\Rightarrow \quad =\tan^{-1}\tan x=x$.

Q05. **a)** We have $\sin^{-1}(1-x)-2\sin^{-1}x=\pi/2$ $\Rightarrow\sin^{-1}(1-x)=\pi/2+2\sin^{-1}x$

$\Rightarrow\sin\left[\sin^{-1}(1-x)\right]=\sin\left[\pi/2+2\sin^{-1}x\right]$ $\Rightarrow 1-x=\cos(2\sin^{-1}x)$

$\Rightarrow 1-x=1-2[\sin(\sin^{-1}x)]^2$ [by using $\cos 2\theta=1-2\sin^2\theta$

$\Rightarrow 1-x=1-2x^2 \qquad \Rightarrow 2x^2-x=0 \qquad \Rightarrow x(2x-1)=0 \qquad \Rightarrow x=0,\ 1/2$

$\because x=1/2$ doesn't satisfy the given equation.

$\therefore x=0$ is the required solution.

b) We have $\sin\left(\sin^{-1}(1/5)+\cos^{-1}x\right)=1 \qquad \Rightarrow \sin^{-1}\sin\left(\sin^{-1}(1/5)+\cos^{-1}x\right)=\sin^{-1}(1)$

$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2} \qquad \therefore x=\frac{1}{5}$ [By comparing to $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$.

c) We've $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4} \qquad \Rightarrow \tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan^{-1}1-\tan^{-1}\left(\frac{x+1}{x+2}\right)$

$\Rightarrow \tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+1\times\left(\frac{x+1}{x+2}\right)}\right) \qquad \Rightarrow \tan\tan^{-1}\left(\frac{x-1}{x-2}\right)=\tan\tan^{-1}\left(\frac{x+2-x-1}{x+2+x+1}\right)$

$\Rightarrow \frac{x-1}{x-2}=\frac{1}{2x+3} \qquad \Rightarrow (2x+3)(x-1)=x-2 \qquad \Rightarrow 2x^2=1 \qquad \therefore x=\pm\frac{1}{\sqrt{2}}$

d) We've $\tan^{-1}\left(\frac{x-1}{x+1}\right)+\tan^{-1}\left(\frac{2x-1}{2x+1}\right)=\tan^{-1}\left(\frac{23}{36}\right) \Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x+1}+\frac{2x-1}{2x+1}}{1-\frac{x-1}{x+1}\times\frac{2x-1}{2x+1}}\right)=\tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \tan\tan^{-1}\left(\frac{2x^2-x-1+2x^2+x-1}{2x^2+3x+1-(2x^2-3x+1)}\right)=\tan\tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \frac{4x^2-2}{6x}=\frac{23}{36} \qquad \Rightarrow 24x^2-23x-12=0$

$\therefore x=\frac{23\pm\sqrt{(-23)^2-4\times 24\times(-12)}}{2\times 24}=\frac{23\pm 41}{2\times 24} \qquad \Rightarrow x=\frac{64}{48},-\frac{18}{48}$ i.e., $\frac{4}{3},-\frac{3}{8}$

$\because x=-\frac{3}{8}$ doesn't satisfy the given equation. $\qquad \therefore x=\frac{4}{3}$ is the required solution.

Alternatively, $\tan^{-1}\left(\frac{x-1}{x+1}\right)+\tan^{-1}\left(\frac{2x-1}{2x+1}\right)=\tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow -\tan^{-1}\left(\frac{1-x}{1+x}\right)-\tan^{-1}\left(\frac{1-2x}{1+2x}\right)=\tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow -\tan^{-1}1+\tan^{-1}x-\tan^{-1}1+\tan^{-1}2x=\tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow -\frac{\pi}{2}+\tan^{-1}\frac{x+2x}{1-2x^2}=\tan^{-1}\left(\frac{23}{36}\right) \qquad \Rightarrow \tan^{-1}\frac{3x}{1-2x^2}=\frac{\pi}{2}+\tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \tan\tan^{-1}\frac{3x}{1-2x^2}=\tan\left(\frac{\pi}{2}+\tan^{-1}\left(\frac{23}{36}\right)\right) \qquad \Rightarrow \frac{3x}{1-2x^2}=-\cot\tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \frac{3x}{1-2x^2}=-\cot\cot^{-1}\left(\frac{36}{23}\right) \qquad \Rightarrow \frac{3x}{2x^2-1}=\frac{36}{23} \qquad \Rightarrow \frac{x}{2x^2-1}=\frac{12}{23}$

$\Rightarrow 24x^2-23x-12=0$. (Now it's same as in the other method given above).

e) We've $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x \qquad \Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x$

$\Rightarrow \frac{\pi}{4} = \frac{3}{2}\tan^{-1} x \qquad \Rightarrow \frac{\pi}{6} = \tan^{-1} x \qquad \Rightarrow \tan\frac{\pi}{6} = \tan\tan^{-1} x \qquad \therefore x = \frac{1}{\sqrt{3}}$

f) We've $\cot^{-1} 2x + \cot^{-1} 3x = \frac{\pi}{4} \qquad \Rightarrow \frac{\pi}{2} - \tan^{-1} 2x + \frac{\pi}{2} - \tan^{-1} 3x = \frac{\pi}{4}$

$\Rightarrow \pi - \frac{\pi}{4} = \tan^{-1} 2x + \tan^{-1} 3x \qquad \Rightarrow \frac{3\pi}{4} = \tan^{-1}\left(\frac{2x+3x}{1-2x.3x}\right)$

$\Rightarrow \tan\frac{3\pi}{4} = \tan\tan^{-1}\left(\frac{5x}{1-6x^2}\right) \qquad \Rightarrow -1 = \frac{5x}{1-6x^2} \qquad \Rightarrow 6x^2 - 5x - 1 = 0$

$\Rightarrow 6x^2 - 5x - 1 = 0 \qquad \Rightarrow (6x+1)(x-1) = 0 \qquad \Rightarrow x = 1, -1/6$

$\because x = -1/6$ doesn't satisfy the given equation. $\therefore x = 1$ is the only required solution.

g) We have $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4} \qquad \Rightarrow \tan^{-1}\frac{2x+3x}{1-2x.3x} = \frac{\pi}{4}$

$\Rightarrow \tan\tan^{-1}\frac{5x}{1-6x^2} = \tan\frac{\pi}{4} \qquad \Rightarrow \frac{5x}{1-6x^2} = 1 \quad \Rightarrow 6x^2 + 5x - 1 = 0 \qquad \Rightarrow x = -1, \frac{1}{6}$

$\because x = -1$ does not satisfy the given equation, $\therefore x = \frac{1}{6}$ is the only required solution.

h) We've $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x) \qquad \Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \tan\tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan\tan^{-1}\left(\frac{2}{\sin x}\right) \qquad \Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$

$\Rightarrow \cos x \sin x - \sin^2 x = 0 \qquad \Rightarrow \sin x(\cos x - \sin x) = 0$

$\Rightarrow \sin x = 0$ or $\cos x = \sin x \qquad \Rightarrow x = 0$ or $x = \frac{\pi}{4}$

As $x = 0$ doesn't satisfy the given equation, so $x = \frac{\pi}{4}$ is the only required solution.

i) Proceed similar as in previous question.

j) Given $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1} x \qquad \Rightarrow \sin^{-1}\left[\frac{3x}{5}\sqrt{1-\left(\frac{4x}{5}\right)^2} + \frac{4x}{5}\sqrt{1-\left(\frac{3x}{5}\right)^2}\right] = \sin^{-1} x$

$\Rightarrow \sin\sin^{-1}\left[\frac{3x}{5}\times\frac{\sqrt{25-16x^2}}{5} + \frac{4x}{5}\times\frac{\sqrt{25-9x^2}}{5}\right] = \sin\sin^{-1} x$

$\Rightarrow 3x\sqrt{25-16x^2} + 4x\sqrt{25-9x^2} = 25x \qquad \Rightarrow x\left[3\sqrt{25-16x^2} + 4\sqrt{25-9x^2} - 25\right] = 0$

$\therefore x = 0$ or $3\sqrt{25-16x^2} + 4\sqrt{25-9x^2} - 25 = 0$

Now $3\sqrt{25-16x^2} = 25 - 4\sqrt{25-9x^2}$ Squaring both the sides, we get :

$9(25-16x^2) = 625 + 16(25-9x^2) - 200\sqrt{25-9x^2} \qquad \Rightarrow 16 = 25 - 9x^2 \qquad \Rightarrow x = \pm 1$

So, $x = 0, \pm 1$ are the required solutions of the given equation.

Alternatively, $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1} x \qquad \Rightarrow \sin\left(\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5}\right) = \sin\sin^{-1} x$

$\Rightarrow \sin\left(\sin^{-1}\frac{3x}{5}\right)\cos\left(\sin^{-1}\frac{4x}{5}\right) + \cos\left(\sin^{-1}\frac{3x}{5}\right)\sin\left(\sin^{-1}\frac{4x}{5}\right) = x$

$\Rightarrow \frac{3x}{5}\sqrt{1-\left(\sin\sin^{-1}\frac{4x}{5}\right)^2} + \sqrt{1-\left(\sin\sin^{-1}\frac{3x}{5}\right)^2}\times\frac{4x}{5} = x$

$\Rightarrow \frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\sqrt{1-\frac{9x^2}{25}}\times\frac{4x}{5}=x$. (Now it's same as in the other method given above).

k) We have $\cos^{-1}x+\sin^{-1}(x/2)=\pi/6$ $\Rightarrow \pi/2-\sin^{-1}x+\sin^{-1}(x/2)=\pi/6$

$\Rightarrow \sin^{-1}x-\sin^{-1}(x/2)=\pi/3$ $\Rightarrow \sin^{-1}\left[x\sqrt{1-\frac{x^2}{4}}-\frac{x}{2}\sqrt{1-x^2}\right]=\frac{\pi}{3}$

$\Rightarrow \sin\sin^{-1}\left[x\sqrt{1-\frac{x^2}{4}}-\frac{x}{2}\sqrt{1-x^2}\right]=\sin\frac{\pi}{3}$ $\Rightarrow x\frac{\sqrt{4-x^2}}{2}-\frac{x}{2}\sqrt{1-x^2}=\frac{\sqrt{3}}{2}$

$\Rightarrow x\sqrt{4-x^2}=\sqrt{3}+x\sqrt{1-x^2}$ $\Rightarrow x^2(4-x^2)=3+x^2(1-x^2)+2x\sqrt{3}\sqrt{1-x^2}$

$\Rightarrow 3x^2-3=2x\sqrt{3}\sqrt{1-x^2}$ $\Rightarrow \sqrt{3}(1-x^2)+2x\sqrt{1-x^2}=0$

$\Rightarrow \left[\sqrt{3}\sqrt{1-x^2}+2x\right]\sqrt{1-x^2}=0$ $\Rightarrow \sqrt{1-x^2}=0$ or $\sqrt{3}\sqrt{1-x^2}+2x=0$

$\Rightarrow 1-x^2=0$ or $\sqrt{3}\sqrt{1-x^2}=-2x$ $\Rightarrow x=\pm1$ or $3-3x^2=4x^2$

$\Rightarrow x=\pm1$ or $x=\pm\sqrt{\frac{3}{7}}$ But $x=-1,\pm\sqrt{\frac{3}{7}}$ doesn't satisfy the given equation.

$\therefore x=1$ is the required solution.

Alternatively, $\cos^{-1}x+\sin^{-1}(x/2)=\pi/6$ $\Rightarrow \sin\sin^{-1}(x/2)=\sin(\pi/6-\cos^{-1}x)$

$\Rightarrow x/2=\sin(\pi/6)\cos(\cos^{-1}x)-\cos(\pi/6)\sin(\cos^{-1}x)$

$\Rightarrow \frac{x}{2}=\frac{1}{2}(x)-\frac{\sqrt{3}}{2}\times\sqrt{1-(\cos\cos^{-1}x)^2}$ $\Rightarrow \sqrt{1-x^2}=0$ $\Rightarrow x^2=1$ $\Rightarrow x=\pm1$

$\because x=-1$ doesn't satisfy the given equation. $\therefore x=1$ is the required solution.

l) Given $\tan^{-1}(x+1)+\cot^{-1}(x-1)=\sin^{-1}(4/5)+\cos^{-1}(3/5)$

$\Rightarrow \tan^{-1}(x+1)+[\pi/2-\tan^{-1}(x-1)]=\tan^{-1}(4/3)+\tan^{-1}(4/3)$

$\Rightarrow \tan^{-1}\left(\frac{x+1-x+1}{1+x^2-1}\right)+\frac{\pi}{2}=\tan^{-1}\left(\frac{2\times\frac{4}{3}}{1-\frac{16}{9}}\right)$ $\Rightarrow \tan\left[\frac{\pi}{2}+\tan^{-1}\left(\frac{2}{x^2}\right)\right]=\tan\tan^{-1}\left(\frac{24}{-7}\right)$

$\Rightarrow -\cot\left[\tan^{-1}\left(\frac{2}{x^2}\right)\right]=-\frac{24}{7}$ $\Rightarrow \cot\left[\cot^{-1}\left(\frac{x^2}{2}\right)\right]=\frac{24}{7}$ $\Rightarrow x=\pm4\sqrt{\frac{3}{7}}$.

m) Given $\tan^{-1}x+2\cot^{-1}x=2\pi/3$ $\Rightarrow \tan^{-1}x+\cot^{-1}x+\cot^{-1}x=2\pi/3$

$\Rightarrow \pi/2+\cot^{-1}x=2\pi/3$ $\Rightarrow \cot^{-1}x=\pi/6$ $\therefore x=\cot(\pi/6)=\sqrt{3}$

n) We've $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$ $\Rightarrow \tan^{-1}\left[\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right]=\tan^{-1}\frac{8}{31}$

$\Rightarrow \tan\tan^{-1}\left[\frac{2x}{2-x^2}\right]=\tan\tan^{-1}\frac{8}{31}$ $\Rightarrow \frac{2x}{2-x^2}=\frac{8}{31}$

$\Rightarrow 4x^2+31x-8=0$ $\Rightarrow (4x-1)(x+8)=0$ $\Rightarrow x=-8,1/4$

As $x=-8$ does not satisfy the given equation so, $x=1/4$ is the only required solution.

o) We have $\tan\cos^{-1}x=\sin\cot^{-1}\frac{1}{2}$ $\Rightarrow \tan\tan^{-1}\frac{\sqrt{1-x^2}}{x}=\sin\sin^{-1}\frac{2}{\sqrt{5}}$

$\Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}}$ $\quad \Rightarrow \frac{1-x^2}{x^2} = \frac{4}{5}$ $\quad \Rightarrow 9x^2 = 5$ $\quad \Rightarrow x = \pm\frac{\sqrt{5}}{3}$

$\because x = -\frac{\sqrt{5}}{3}$ does not satisfy the given equation so, $x = \frac{\sqrt{5}}{3}$ is the only required solution.

p) We have $\sin\left[2\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right] = 1$ $\Rightarrow 2\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\} = \sin^{-1}(1) = \frac{\pi}{2}$

$\Rightarrow \cot\left(2\tan^{-1}x\right) = \cos\frac{\pi}{4}$ $\quad \Rightarrow \cot\left[\tan^{-1}\frac{2x}{1-x^2}\right] = \frac{1}{\sqrt{2}}$ $\quad \Rightarrow \cot\left[\cot^{-1}\frac{1-x^2}{2x}\right] = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{1-x^2}{2x} = \frac{1}{\sqrt{2}}$ $\quad \therefore x = \frac{-1\pm\sqrt{3}}{\sqrt{2}}$.

q) We have $\sec^2\tan^{-1}2 + \operatorname{cosec}^2\cot^{-1}3 = x$ $\quad \Rightarrow 1+\left(\tan\tan^{-1}2\right)^2 + 1 + \left(\cot\cot^{-1}3\right)^2 = x$

$\Rightarrow 1+(2)^2+1+(3)^2 = x$ $\quad \therefore x = 15$

r) We have $\tan^{-1}4 + \cot^{-1}x = \pi/2$ $\quad$ By comparing to $\tan^{-1}x + \cot^{-1}x = \pi/2$, we get $x = 4$.

s) We've $\tan^{-1}x + \tan^{-1}(1-x) = \cot^{-1}\frac{7}{9}$ $\quad \Rightarrow \tan^{-1}\frac{x+1-x}{1-x(1-x)} = \tan^{-1}\frac{9}{7}$

$\Rightarrow \tan\tan^{-1}\frac{1}{1-x+x^2} = \tan\tan^{-1}\frac{9}{7}$ $\quad \Rightarrow \frac{1}{1-x+x^2} = \frac{9}{7}$ $\quad \Rightarrow 9x^2 - 9x + 2 = 0$

$\Rightarrow (3x-2)(3x-1) = 0$ $\quad \therefore x = 1/3,\ 2/3$

t) We have $\cos\left(2\sin^{-1}x\right) = 1/9$ $\quad \Rightarrow 1 - 2(\sin\sin^{-1}x)^2 = 1/9$

$\Rightarrow 1 - 1/9 = 2x^2$ $\quad \therefore x = \pm\frac{2}{3}$.

u) We have $3\sin^{-1}\left[\frac{2x}{1+x^2}\right] - 4\cos^{-1}\left[\frac{1-x^2}{1+x^2}\right] + 2\tan^{-1}\left[\frac{2x}{1-x^2}\right] = \frac{\pi}{3}$

$\Rightarrow 3\times 2\tan^{-1}x - 4\times 2\tan^{-1}x + 2\times 2\tan^{-1}x = \frac{\pi}{3}$ $\quad \Rightarrow 2\tan^{-1}x = \frac{\pi}{3}$

$\Rightarrow x = \tan\frac{\pi}{6}$ $\quad \therefore x = \frac{1}{\sqrt{3}}$

v) Given $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$

$\Rightarrow \tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$ $\quad \Rightarrow \tan^{-1}\frac{(x-1)+(x+1)}{1-(x-1)(x+1)} = \tan^{-1}\frac{3x-x}{1+3x.x}$

$\Rightarrow \tan\tan^{-1}\frac{2x}{2-x^2} = \tan\tan^{-1}\frac{2x}{1+3x^2}$ $\quad \Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2}$

$\Rightarrow x\left[1+3x^2-2+x^2\right] = 0$ $\quad \Rightarrow x = 0$ or $4x^2 - 1 = 0$ $\quad \therefore x = 0, \pm 1/2$

w) We have $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$ $\quad \Rightarrow \sin^{-1}(1-x) = \pi/2 - \sin^{-1}x - \sin^{-1}x$

$\Rightarrow \sin\sin^{-1}(1-x) = \sin\left[\pi/2 - 2\sin^{-1}x\right]$ $\quad \Rightarrow 1-x = \cos(2\sin^{-1}x)$

$\Rightarrow 1-x = 1-2(\sin\sin^{-1}x)^2$ $\quad \Rightarrow 2x^2 - x = 0$ $\quad \Rightarrow x(2x-1) = 0$ $\quad \therefore x = 0, 1/2$

x) We have $\tan^{-1}(2-x) + \tan^{-1}(2+x) = \tan^{-1}\frac{2}{3}$ $\quad \Rightarrow \tan^{-1}\frac{(2-x)+(2+x)}{1-(2-x)(2+x)} = \tan^{-1}\frac{2}{3}$

$\Rightarrow \tan\tan^{-1}\frac{4}{x^2-3} = \tan\tan^{-1}\frac{2}{3}$ $\quad \Rightarrow \frac{4}{x^2-3} = \frac{2}{3}$ $\quad \Rightarrow x^2 = 9$ $\quad \therefore x = \pm 3$

y) We have $\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$ $\Rightarrow \sin^{-1}\left(\frac{5}{x}\right)=\frac{\pi}{2}-\sin^{-1}\left(\frac{12}{x}\right)$

$\Rightarrow \sin^{-1}\left(\frac{5}{x}\right)=\cos^{-1}\left(\frac{12}{x}\right)$ $\Rightarrow \sin\sin^{-1}\left(\frac{5}{x}\right)=\sin\cos^{-1}\left(\frac{12}{x}\right)$

$\Rightarrow \frac{5}{x}=\sqrt{1-\left[\cos\cos^{-1}\left(\frac{12}{x}\right)\right]^2}$ $\Rightarrow \frac{25}{x^2}=1-\frac{144}{x^2}$ $\Rightarrow \frac{169}{x^2}=1$ $\Rightarrow x=\pm 13$

As $x=-13$ doesn't satisfy the given equation so, $x=13$ is the only required solution.

z) We've $\sin[\cot^{-1}(x+1)]=\cos(\tan^{-1}x)$ $\Rightarrow \sin\sin^{-1}\left(\frac{1}{\sqrt{1+(x+1)^2}}\right)=\cos\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$\Rightarrow \frac{1}{\sqrt{1+(x+1)^2}}=\frac{1}{\sqrt{1+x^2}}$ $\Rightarrow 2+x^2+2x=1+x^2$ $\therefore x=-\frac{1}{2}$.

aa) Here $\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$ $\Rightarrow \tan^{-1}\left(\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\times\frac{x-1}{x}}\right)=\tan^{-1}(-7)$

$\Rightarrow \tan^{-1}\left(\frac{x^2+x+x^2-2x+1}{x^2-x-x^2+1}\right)=\tan^{-1}(-7)$ $\Rightarrow \left(\frac{2x^2-x+1}{-x+1}\right)=-7$

$\Rightarrow 2x^2-x+1=7x-7$ $\Rightarrow 2x^2-8x+8=0$ or, $x^2-4x+4=0$

$\Rightarrow (x-2)^2=0$ $\Rightarrow x=2$.

Since $x=2$ doesn't satisfy the given equation so, the given equation has no solution.

Alternatively, $\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$

$\Rightarrow -\tan^{-1}\left(\frac{1+x}{1-x}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$

$\Rightarrow -\tan^{-1}1-\tan^{-1}x+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$

$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x}-x}{1+\frac{x-1}{x}\times x}\right)=\tan^{-1}(-7)+\frac{\pi}{4}$

$\Rightarrow \tan\tan^{-1}\left(\frac{\frac{x-1-x^2}{x}}{\frac{x+x^2-x}{x}}\right)=\tan\left(\tan^{-1}(-7)+\frac{\pi}{4}\right)$

$\Rightarrow \left(\frac{x-1-x^2}{x+x^2-x}\right)=\frac{\tan\tan^{-1}(-7)+\tan\frac{\pi}{4}}{1-\tan\tan^{-1}(-7)\tan\frac{\pi}{4}}$ $\Rightarrow \left(\frac{x-1-x^2}{x^2}\right)=\frac{-7+1}{1-(-7)}$

$\Rightarrow \frac{-x+1+x^2}{x^2}=\frac{3}{4}$ $\Rightarrow x^2-4x+4=0$. (Now it's same as in the other method given above).

ab) We have $\tan^{-1}\left(\frac{x-5}{x-6}\right)+\tan^{-1}\left(\frac{x+5}{x+6}\right)=\frac{\pi}{4}$ $\Rightarrow \tan^{-1}\left(\frac{\frac{x-5}{x-6}+\frac{x+5}{x+6}}{1-\frac{x-5}{x-6}\times\frac{x+5}{x+6}}\right)=\frac{\pi}{4}$

$\Rightarrow \tan\tan^{-1}\left(\frac{(x-5)(x+6)+(x-6)(x+5)}{x^2-36-x^2+25}\right)=\tan\frac{\pi}{4}$ $\Rightarrow \frac{x^2+x-30+x^2-x-30}{-11}=1$

$\Rightarrow 2x^2=49$ $\therefore x=\pm\frac{7}{\sqrt{2}}$ or $\pm\frac{7}{2}\sqrt{2}$.

ac) We've $\tan^{-1}\left(\frac{x-2}{x-3}\right)+\tan^{-1}\left(\frac{x+2}{x+3}\right)=\frac{\pi}{4}, |x|<1$ $\Rightarrow \tan^{-1}\left(\frac{\frac{x-2}{x-3}+\frac{x+2}{x+3}}{1-\frac{x-2}{x-3}\times\frac{x+2}{x+3}}\right)=\frac{\pi}{4}$

$\Rightarrow \tan\tan^{-1}\left(\frac{x^2+x-6+x^2-x-6}{x^2-9-x^2+4}\right)=\tan\frac{\pi}{4}$ $\Rightarrow \frac{2x^2-12}{-5}=1$

$\Rightarrow x^2=\frac{7}{2}$ $\Rightarrow x=\pm\sqrt{\frac{7}{2}}$ $\because |x|<1$ so, the given equation has no solution.

ad) Here $\tan^{-1}\left(\frac{x-2}{x-1}\right)+\tan^{-1}\left(\frac{x+2}{x+1}\right)=\frac{\pi}{4}$ $\Rightarrow \tan^{-1}\left(\frac{\frac{x-2}{x-1}+\frac{x+2}{x+1}}{1-\frac{x-2}{x-1}\times\frac{x+2}{x+1}}\right)=\frac{\pi}{4}$

$\Rightarrow \tan\tan^{-1}\left(\frac{(x-2)(x+1)+(x+2)(x-1)}{(x^2-1)-(x^2-4)}\right)=\tan\frac{\pi}{4}$ $\Rightarrow \left(\frac{x^2-x-2+x^2+x-2}{3}\right)=1$

$\Rightarrow \frac{2x^2-4}{3}=1$ $\Rightarrow x^2=\frac{7}{2}$ $\therefore x=\pm\sqrt{\frac{7}{2}}$.

ae) Given that $\tan^{-1}\left(\frac{2-x}{2+x}\right)=\frac{1}{2}\tan^{-1}\frac{x}{2}, x>0$

$\Rightarrow 2\tan^{-1}\left(\frac{2-x}{2+x}\right)=\tan^{-1}\frac{x}{2}$ $\Rightarrow \tan^{-1}\frac{2\left(\frac{2-x}{2+x}\right)}{1-\left(\frac{2-x}{2+x}\right)^2}=\tan^{-1}\frac{x}{2}$

$\Rightarrow \tan^{-1}\frac{2(2-x)(2+x)}{(2+x)^2-(2-x)^2}=\tan^{-1}\frac{x}{2}$ $\Rightarrow \tan^{-1}\frac{2(4-x^2)}{4x}=\tan^{-1}\frac{x}{2}$

$\Rightarrow \tan^{-1}\frac{4-x^2}{2x}=\tan^{-1}\frac{x}{2}$ $\Rightarrow \frac{4-x^2}{2x}=\frac{x}{2}$ $\Rightarrow 4-x^2=x^2$ $\therefore x=\sqrt{2}$ [$\because x>0$.

Q06. **a)** We have $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\frac{2\pi}{3}+\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)$

$\Rightarrow \quad =\frac{2\pi}{3}+\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)=\frac{2\pi}{3}+\frac{\pi}{3}=\pi$.

b) Given $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)=\cos^{-1}\left(\cos\left(2\pi-\frac{\pi}{3}\right)\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$

c) We have $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{6}\right)\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}$

d) Given $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left(\tan\left(\pi+\frac{\pi}{6}\right)\right)=\tan^{-1}\left(\tan\frac{\pi}{6}\right)=\frac{\pi}{6}$

e) Given $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)=\sin^{-1}\left(\sin\left(\pi-\frac{2\pi}{5}\right)\right)=\sin^{-1}\left(\sin\left(\frac{2\pi}{5}\right)\right)=\frac{2\pi}{5}$

f) We've $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{5}\right)\right)=\sin^{-1}\left(\sin\frac{\pi}{5}\right)=\frac{\pi}{5}$

g) We've $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{3\pi}{4}\right)=\operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\pi-\frac{\pi}{4}\right)\right)=\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)=\frac{\pi}{4}$

h) We've $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)=\tan^{-1}\left(\tan\left(\pi-\frac{\pi}{6}\right)\right)=\tan^{-1}\left(-\tan\frac{\pi}{6}\right)=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)=-\frac{\pi}{6}$

i) We've $\cos^{-1}\left(\cos\frac{7\pi}{5}\right)=\cos^{-1}\left(\cos\left(\pi+\frac{2\pi}{5}\right)\right)=\cos^{-1}\left(-\cos\left(\frac{2\pi}{5}\right)\right)$

$\Rightarrow \quad =\pi-\cos^{-1}\left(\cos\left(\frac{2\pi}{5}\right)\right)=\pi-\frac{2\pi}{5}=\frac{3\pi}{5}$

j) Given $\sin^{-1}(\sin 2)=\sin^{-1}\sin(\pi-2)=\pi-2$

[As $2\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ i.e., $2\notin[-1.57,1.57]$, $\therefore \sin^{-1}(\sin 2)\neq 2$. Whereas $\pi-2\in[-1.57,1.57]$]

k) Given $\sin^{-1}(\sin 10)=\sin^{-1}\sin(3\pi-10)=3\pi-10$

l) We have $\cos^{-1}\cos 320^\circ-\sin^{-1}\sin 320^\circ=\cos^{-1}\cos(360^\circ-40^\circ)-\sin^{-1}\sin(360^\circ-40^\circ)$

$\Rightarrow \quad =\cos^{-1}\cos 40^\circ-\sin^{-1}[-\sin 40^\circ]=40^\circ+\sin^{-1}\sin 40^\circ=80^\circ$

Q07. **a)** $\cot\left(\tan^{-1}a+\cot^{-1}a\right)=\cot\left(\frac{\pi}{2}\right)=0$

b) $\cos\left(\sec^{-1}x+\operatorname{cosec}^{-1}x\right)=\cos\frac{\pi}{2}=0$

c) $\tan\left[2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right]=\tan\left[\tan^{-1}\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right)-\frac{\pi}{4}\right]=\tan\left[\tan^{-1}\frac{5}{12}-\frac{\pi}{4}\right]$

$\Rightarrow \quad =\frac{\tan\tan^{-1}\left(\frac{5}{12}\right)-\tan\frac{\pi}{4}}{1+\tan\tan^{-1}\left(\frac{5}{12}\right)\tan\frac{\pi}{4}}=\frac{\frac{5}{12}-1}{1+\frac{5}{12}}=-\frac{7}{17}$

d) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]=\sin\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right]=\sin\frac{\pi}{2}=1$

e) $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]=\tan\frac{1}{2}\left[2\tan^{-1}x+2\tan^{-1}y\right]$

$\Rightarrow \quad =\tan\left[\tan^{-1}x+\tan^{-1}y\right]=\frac{x+y}{1-xy}$

f) $\tan^{-1}[2\cos(2\sin^{-1}(0.5))]=\tan^{-1}[2\cos 2\sin^{-1}\frac{1}{2}]=\tan^{-1}\left[2\cos 2\left(\frac{\pi}{6}\right)\right]$

$\Rightarrow \quad =\tan^{-1}\left[2\cos\left(\frac{\pi}{3}\right)\right]=\tan^{-1}\left[2\left(\frac{1}{2}\right)\right]=\tan^{-1}1=\frac{\pi}{4}$

g) $\tan[2\tan^{-1}(0.2)]=\tan\left[2\tan^{-1}\frac{1}{5}\right]=\tan\left[\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}\right]=\tan\left[\tan^{-1}\frac{5}{12}\right]=\frac{5}{12}$

h) $\cot^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]=\cot^{-1}[-1]=\cot^{-1}\left[\cot\left(\frac{3\pi}{4}\right)\right]=\frac{3\pi}{4}$

i) $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)=\frac{\tan\tan^{-1}\frac{3}{4}+\tan\tan^{-1}\frac{2}{3}}{1-\tan\tan^{-1}\frac{3}{4}\times\tan\tan^{-1}\frac{2}{3}}=\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}=\frac{17}{6}.$

j) $\sin\left(2\cos^{-1}\left(-\frac{3}{5}\right)\right)=2\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)\cos\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$

$\Rightarrow \quad =2\sin\left(\pi-\cos^{-1}\left(\frac{3}{5}\right)\right)\cos\left(\pi-\cos^{-1}\left(\frac{3}{5}\right)\right)$

$\Rightarrow \quad =2\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)\left(-\cos\cos^{-1}\left(\frac{3}{5}\right)\right)$

$\Rightarrow \quad =2\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right)\left(-\left(\frac{3}{5}\right)\right)=2\left(\frac{4}{5}\right)\left(-\left(\frac{3}{5}\right)\right)=-\frac{24}{25}$

Q08. **a)** $\text{LHS}:\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{4}{3}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right)=\tan^{-1}\frac{63}{16}=\text{RHS}.$

b) Given $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ $\quad\Rightarrow 2\tan^{-1}\left(\frac{1}{4}\right)+2\tan^{-1}\left(\frac{2}{9}\right)=\cos^{-1}\left(\frac{3}{5}\right)$

$\text{LHS}:2\tan^{-1}\left(\frac{1}{4}\right)+2\tan^{-1}\left(\frac{2}{9}\right)=\tan^{-1}\left(\frac{\frac{2}{4}}{1-\frac{1}{16}}\right)+\tan^{-1}\left(\frac{\frac{4}{9}}{1-\frac{4}{81}}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\frac{8}{15}\right)+\tan^{-1}\left(\frac{36}{77}\right)=\tan^{-1}\left(\frac{\frac{8}{15}+\frac{36}{77}}{1-\frac{8}{15}\times\frac{36}{77}}\right)=\tan^{-1}\left(\frac{616+540}{1155-288}\right)=\tan^{-1}\frac{1156}{867}$

$\Rightarrow \quad =\tan^{-1}\frac{289\times4}{289\times3}=\cos^{-1}\frac{3}{5}=\text{RHS}.$

c) $\text{LHS}:\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)=\tan^{-1}(1)+\left(\frac{\pi}{2}-\cot^{-1}2\right)+\left(\frac{\pi}{2}-\cot^{-1}3\right)$

$$\Rightarrow \quad = \pi + \tan^{-1}(1) - \left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right) = \pi + \tan^{-1}(1) - \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$$

$$\Rightarrow \quad = \pi + \tan^{-1}(1) - \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \pi + \tan^{-1}(1) - \tan^{-1}(1) = \pi = \text{RHS}$$

d) $\text{LHS}: \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}}\right)$

$$\Rightarrow \quad = \tan^{-1}\left(\frac{15}{20}\right) = \tan^{-1}\frac{3}{4} = \text{RHS}$$

e) $\text{LHS}: 2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{\frac{2}{2}}{1-\frac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$$\Rightarrow \quad = \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{31}{17}\right) = \text{RHS}$$

f) $\text{LHS}: \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \tan^{-1}\frac{3}{4} - \tan^{-1}\frac{8}{15} = \tan^{-1}\left(\frac{\frac{3}{4}-\frac{8}{15}}{1+\frac{3}{4}\times\frac{8}{15}}\right)$

$$\Rightarrow \quad = \tan^{-1}\frac{13}{84} = \cos^{-1}\frac{84}{85} = \text{RHS}$$

g) LHS: Let $Y = \sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \tan^{-1}\frac{12}{5} + \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{63}{16}$

$\Rightarrow \quad = \tan^{-1}\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}} + \tan^{-1}\frac{63}{16} \qquad \Rightarrow Y = \tan^{-1}\left(-\frac{63}{16}\right) + \tan^{-1}\frac{63}{16}$ ...(i)

Let $\tan^{-1}\left(-\frac{63}{16}\right) = x \qquad \Rightarrow \tan x = -\frac{63}{16} \qquad \Rightarrow -\tan x = \frac{63}{16} \qquad \Rightarrow \tan(\pi - x) = \frac{63}{16}$

$\Rightarrow \pi - x = \tan^{-1}\frac{63}{16} \qquad \Rightarrow \pi = \tan^{-1}\frac{63}{16} + x \qquad \Rightarrow \pi = \tan^{-1}\frac{63}{16} + \tan^{-1}\left(-\frac{63}{16}\right)$

$\therefore Y = \pi = \text{RHS}$ [By using (i)

h) Given $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} \qquad \Rightarrow \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} + \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9\pi}{8}$

$\text{LHS}: \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} + \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\left(\sin^{-1}\frac{2\sqrt{2}}{3} + \sin^{-1}\frac{1}{3}\right) = \frac{9}{4}\left(\sin^{-1}\left[\frac{2\sqrt{2}}{3}\sqrt{1-\frac{1}{9}} + \frac{1}{3}\sqrt{1-\frac{8}{9}}\right]\right)$

$\Rightarrow \quad = \frac{9}{4}\left(\sin^{-1}\left[\frac{2\sqrt{2}}{3}\times\frac{2\sqrt{2}}{3}+\frac{1}{3}\times\frac{1}{3}\right]\right) = \frac{9}{4}\left(\sin^{-1}\left[\frac{9}{9}\right]\right) = \frac{9}{4}\sin^{-1}(1)$

$\Rightarrow \quad = \frac{9}{4}\times\frac{\pi}{2} = \frac{9\pi}{8} = \text{RHS}$

i) LHS: $2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4} \quad \Rightarrow \quad = \tan^{-1}\left(\frac{\frac{6}{4}}{1-\frac{9}{16}}\right) \quad \Rightarrow \quad = \tan^{-1}\left(\frac{24}{7}\right) = \text{RHS}$

j) LHS: $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{3}{4} = \tan^{-1}\left(\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\times\frac{3}{4}}\right)$

$\Rightarrow \quad = \tan^{-1}\left(\frac{56}{33}\right) = \sin^{-1}\left(\frac{56}{65}\right) = \text{RHS}.$

k) LHS: $\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$

$\Rightarrow \quad = \tan^{-1}\left(\frac{12}{34}\right)+\tan^{-1}\left(\frac{11}{23}\right) = \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$

$\Rightarrow \quad = \tan^{-1}\left(\frac{325}{391-66}\right) = \tan^{-1}(1) = \frac{\pi}{4} = \text{RHS}$

l) LHS: $\cot^{-1}\left[2\tan\left(\cos^{-1}\frac{8}{17}\right)\right]+\tan^{-1}\left[2\tan\left(\sin^{-1}\frac{8}{17}\right)\right]$

$\Rightarrow \quad = \cot^{-1}\left[2\tan\left(\tan^{-1}\frac{15}{8}\right)\right]+\tan^{-1}\left[2\tan\left(\tan^{-1}\frac{8}{15}\right)\right]$

$\Rightarrow \quad = \cot^{-1}\left[2\times\frac{15}{8}\right]+\tan^{-1}\left[2\times\frac{8}{15}\right] = \cot^{-1}\left[\frac{15}{4}\right]+\tan^{-1}\left[\frac{16}{15}\right] = \tan^{-1}\left[\frac{4}{15}\right]+\tan^{-1}\left[\frac{16}{15}\right]$

$\Rightarrow \quad = \tan^{-1}\left[\frac{\frac{4}{15}+\frac{16}{15}}{1-\frac{4}{15}\times\frac{16}{15}}\right] = \tan^{-1}\left[\frac{60+240}{225-64}\right] = \tan^{-1}\left[\frac{300}{161}\right] = \text{RHS}$

m) Consider $\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3) = \tan^{-1}(1)+\left(\frac{\pi}{2}-\cot^{-1}2\right)+\left(\frac{\pi}{2}-\cot^{-1}3\right)$

$\Rightarrow \quad = \pi+\tan^{-1}(1)-\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right) = \pi+\tan^{-1}(1)-\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$

$\Rightarrow \quad = \pi+\tan^{-1}(1)-\tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \pi+\tan^{-1}(1)-\tan^{-1}(1) = \pi \qquad$ Proved.

Also consider $2\left(\tan^{-1}(1)+\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)\right)=2\left(\tan^{-1}\left(\tan\frac{\pi}{4}\right)+\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)\right)$

$\Rightarrow \quad =2\left(\frac{\pi}{4}+\tan^{-1}\left(\frac{5/6}{5/6}\right)\right)=2\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\pi$ Proved.

Hence $\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)=\pi=2\left(\tan^{-1}(1)+\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)\right)$.

n) LHS : $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right)+\tan^{-1}\left(\frac{1}{8}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right)=\tan^{-1}\left(\frac{65}{65}\right)=\tan^{-1}1=\frac{\pi}{4}=\text{RHS}$

o) LHS : $2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\tan^{-1}\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right)+\tan^{-1}\frac{1}{7}+\tan^{-1}\left(\frac{\frac{2}{8}}{1-\frac{1}{64}}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\frac{1}{7}+\tan^{-1}\left(\frac{16}{63}\right)=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{1}{7}}{1-\frac{5}{12}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{16}{63}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\frac{47}{79}\right)+\tan^{-1}\left(\frac{16}{63}\right)=\tan^{-1}\left(\frac{\frac{47}{79}+\frac{16}{63}}{1-\frac{47}{79}\times\frac{16}{63}}\right)=\tan^{-1}\left(\frac{4225}{4225}\right)=\tan^{-1}1=\frac{\pi}{4}=\text{RHS}$

p) LHS : $\cot^{-1}9+\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}=\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{4}{5}$

$\Rightarrow \quad =\tan^{-1}\left(\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\times\frac{4}{5}}\right)=\tan^{-1}\left(\frac{41}{41}\right)=\tan^{-1}1=\frac{\pi}{4}=\text{RHS}$

q) LHS : $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=2\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{17}{31}=\tan^{-1}\left(\frac{\frac{6}{4}}{1-\frac{9}{16}}\right)-\tan^{-1}\frac{17}{31}$

$\Rightarrow \quad =\tan^{-1}\frac{24}{7}-\tan^{-1}\frac{17}{31}=\tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)=\tan^{-1}\left(\frac{625}{625}\right)=\tan^{-1}1=\frac{\pi}{4}=\text{RHS}$

Note Q08 (r) and (s) have been left intentionally for you.

t) LHS : $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right)$

$\Rightarrow \quad =\tan^{-1}\left(\frac{31}{17}\right)=\sin^{-1}\frac{31}{25\sqrt{2}}=\text{RHS}$

Q09. Given $y=\cot^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)=\frac{\pi}{2}-\tan^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)$

$\Rightarrow y=\frac{\pi}{2}-2\tan^{-1}\left(\sqrt{\cos x}\right)=\frac{\pi}{2}-\cos^{-1}\left\{\frac{1-\left(\sqrt{\cos x}\right)^2}{1+\left(\sqrt{\cos x}\right)^2}\right\}=\frac{\pi}{2}-\cos^{-1}\left\{\frac{1-\cos x}{1+\cos x}\right\}$

$\Rightarrow y=\frac{\pi}{2}-\cos^{-1}\left(\tan^2\frac{x}{2}\right)$ $\quad\Rightarrow \cos^{-1}\left(\tan^2\frac{x}{2}\right)=\frac{\pi}{2}-y$

$\Rightarrow \tan^2\frac{x}{2}=\cos\left(\frac{\pi}{2}-y\right)$ $\quad\therefore \tan^2\frac{x}{2}=\sin y$. Hence proved.

Q10. Given $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ $\quad\Rightarrow \cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$

$\Rightarrow \cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}=\cos^{-1}(-z)$ $\quad\Rightarrow xy-\sqrt{1-x^2}\sqrt{1-y^2}=-z$

$\Rightarrow xy+z=\sqrt{1-x^2}\sqrt{1-y^2}$ $\quad\Rightarrow x^2y^2+2xyz+z^2=(1-x^2)(1-y^2)$

$\Rightarrow x^2+y^2+z^2+2xyz=1$. Hence proved.

Q11. We have $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$ $\quad\Rightarrow \tan^{-1}\frac{x+y}{1-xy}=\frac{\pi}{4}$ $\quad\Rightarrow \tan\tan^{-1}\frac{x+y}{1-xy}=\tan\frac{\pi}{4}$

$\Rightarrow \frac{x+y}{1-xy}=1$ $\quad\therefore x+y+xy=1$

Q12. LHS: $\cos^{-1}\frac{1}{5\sqrt{2}}=\alpha,\ \sin^{-1}\frac{4}{\sqrt{17}}=\beta$ $\quad\Rightarrow \cos\alpha=\frac{1}{5\sqrt{2}},\ \sin\beta=\frac{4}{\sqrt{17}}$ $\quad\therefore \tan\alpha=7,\tan\beta=4$

Now $\tan\left[\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{17}}\right]=\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ $\quad\Rightarrow \quad =\frac{7-4}{1+7\times4}=\frac{3}{29}$

Q13. Given $\lambda=\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ $\quad$ Put $x^2=\cos2\theta\ldots$(i)

$\therefore \lambda=\tan^{-1}\left(\frac{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}{\sqrt{1+\cos2\theta}+\sqrt{1-\cos2\theta}}\right)$ $\quad\Rightarrow \lambda=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$

$\Rightarrow \lambda=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)=\tan^{-1}\frac{1-\tan\theta}{1+\tan\theta}=\tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$

$\Rightarrow \lambda=\frac{\pi}{4}-\theta$ $\quad\Rightarrow 2\lambda=\frac{\pi}{2}-2\theta$ $\quad\Rightarrow 2\lambda=\frac{\pi}{2}-\cos^{-1}x^2$

$\Rightarrow \cos^{-1}x^2=\frac{\pi}{2}-2\lambda$ $\quad\Rightarrow \cos\cos^{-1}x^2=\cos\left(\frac{\pi}{2}-2\lambda\right)$ $\quad\therefore x^2=\sin2\lambda$.

Q14. **(a)** LHS : $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)=\cos^{-1}\left(\frac{1-\frac{a-b}{a+b}\tan^2\frac{\theta}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{\theta}{2}}\right)$

$$\Rightarrow \quad =\cos^{-1}\left(\frac{(a+b)-(a-b)\tan^2\frac{\theta}{2}}{(a+b)+(a-b)\tan^2\frac{\theta}{2}}\right)=\cos^{-1}\left(\frac{a\left(1-\tan^2\frac{\theta}{2}\right)+b\left(1+\tan^2\frac{\theta}{2}\right)}{a\left(1+\tan^2\frac{\theta}{2}\right)+b\left(1-\tan^2\frac{\theta}{2}\right)}\right)$$

$$\Rightarrow \quad =\cos^{-1}\left(\frac{a\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)+b}{a+b\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)}\right)=\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)=\text{RHS}.$$

(b) RHS: $2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)=\cos^{-1}\left(\frac{1-\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)^2}{1+\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)^2}\right)$

$$\Rightarrow \quad =\cos^{-1}\left(\frac{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}-\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}}{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}}\right)$$

$$\Rightarrow \quad =\cos^{-1}\left(\frac{\left(2\cos^2\frac{\alpha}{2}\right)\left(2\cos^2\frac{\beta}{2}\right)-\left(2\sin^2\frac{\alpha}{2}\right)\left(2\sin^2\frac{\beta}{2}\right)}{\left(2\cos^2\frac{\alpha}{2}\right)\left(2\cos^2\frac{\beta}{2}\right)+\left(2\sin^2\frac{\alpha}{2}\right)\left(2\sin^2\frac{\beta}{2}\right)}\right)$$

$$\Rightarrow \quad =\cos^{-1}\left(\frac{(1+\cos\alpha)(1+\cos\beta)-(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)+(1-\cos\alpha)(1-\cos\beta)}\right)$$

$$\Rightarrow \quad =\cos^{-1}\left(\frac{1+\cos\alpha+\cos\beta+\cos\alpha\cos\beta-(1-\cos\alpha-\cos\beta+\cos\alpha\cos\beta)}{1+\cos\alpha+\cos\beta+\cos\alpha\cos\beta+1-\cos\alpha-\cos\beta+\cos\alpha\cos\beta}\right)$$

$$\Rightarrow \quad =\cos^{-1}\left(\frac{2\cos\alpha+2\cos\beta}{2+2\cos\alpha\cos\beta}\right)=\cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right)=\text{RHS}.$$

Q15. **(a)** $\tan(\cos^{-1}x)=\sin(\tan^{-1}2)$ $\qquad \Rightarrow \tan\tan^{-1}\frac{\sqrt{1-x^2}}{x}=\sin\sin^{-1}\frac{2}{\sqrt{5}}$

$\Rightarrow \frac{\sqrt{1-x^2}}{x}=\frac{2}{\sqrt{5}}$ $\qquad \Rightarrow \frac{1-x^2}{x^2}=\frac{4}{5}$ $\qquad \Rightarrow 9x^2=5$ $\qquad \Rightarrow x=\pm\frac{\sqrt{5}}{3}$

$\because x=-\frac{\sqrt{5}}{3}$ does not satisfy the given equation so, $x=\frac{\sqrt{5}}{3}$ is the only required solution.

(b) $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$ $\qquad \Rightarrow \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{-(1-x^2)}\right)=\frac{2\pi}{3}$

$\Rightarrow \cos^{-1}\left(-\frac{1-x^2}{1+x^2}\right)-\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$ $\quad\Rightarrow \pi-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)-2\tan^{-1}x=\frac{2\pi}{3}$

$\Rightarrow \pi-2\tan^{-1}x-2\tan^{-1}x=\frac{2\pi}{3}$ $\quad\Rightarrow \pi-\frac{2\pi}{3}=4\tan^{-1}x$ $\quad\Rightarrow x=\tan\frac{\pi}{12}$

$$\Rightarrow x=\tan\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\frac{\tan\frac{\pi}{3}-\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{3}\times\tan\frac{\pi}{4}}=\frac{\sqrt{3}-1}{1-\sqrt{3}\times 1}=2-\sqrt{3}$$

Q16. Following table lists the range of inverse trigonometric functions other than their principal branch :

Inverse Trigonometric Functions *i.e.* f (x)	**Range of f (x) other than Principal branch**
$\sin^{-1}x$	$\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right],\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ etc.
$\cos^{-1}x$	$[-\pi,0],[\pi,2\pi]$ etc.
$\text{cosec}^{-1}x$	$\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right]-\{-\pi\},\left[\frac{\pi}{2},\frac{3\pi}{2}\right]-\{\pi\}$ etc.
$\sec^{-1}x$	$[-\pi,0]-\left\{-\frac{\pi}{2}\right\},[\pi,2\pi]-\left\{\frac{3\pi}{2}\right\}$ etc.
$\tan^{-1}x$	$\left(-\frac{3\pi}{2},-\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ etc.
$\cot^{-1}x$	$(-\pi,0),(\pi,2\pi)$ etc.

Q17. **(a)** Given $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$ $\quad\Rightarrow \tan^{-1}\frac{x+y}{1-xy}=\frac{\pi}{2}-\tan^{-1}z$

$\Rightarrow \tan\tan^{-1}\frac{x+y}{1-xy}=\tan\left(\frac{\pi}{2}-\tan^{-1}z\right)$ $\quad\Rightarrow \frac{x+y}{1-xy}=\cot\left(\tan^{-1}z\right)$

$\Rightarrow \frac{x+y}{1-xy}=\cot\left(\cot^{-1}\frac{1}{z}\right)$ $\quad\Rightarrow \frac{x+y}{1-xy}=\frac{1}{z}$

$\Rightarrow yz+zx=1-xy$ $\quad\therefore xy+yz+zx=1$. Hence proved.

(b) Given that $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$ $\quad\Rightarrow \tan^{-1}\frac{x+y}{1-xy}=\pi-\tan^{-1}z$

$\Rightarrow \tan\tan^{-1}\frac{x+y}{1-xy}=\tan(\pi-\tan^{-1}z)$ $\quad\Rightarrow \frac{x+y}{1-xy}=-\tan(\tan^{-1}z)$ $\quad\Rightarrow \frac{x+y}{1-xy}=-z$

$\Rightarrow x+y=-z+xyz$ $\quad\therefore x+y+z=xyz$. Hence proved.

Q18. RHS : Let $y=2\tan^{-1}\left[\text{cosec}\tan^{-1}x-\tan\cot^{-1}x\right]$

$$\Rightarrow \quad =2\tan^{-1}\left[\text{cosec}\left(\text{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}\right)-\tan\tan^{-1}\frac{1}{x}\right]$$

$$\Rightarrow \quad =2\tan^{-1}\left[\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right]=2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$$ $\left[\begin{array}{l}\text{Put } x=\tan\theta \\ \Rightarrow \theta=\tan^{-1}x\end{array}\right.\ldots(i)$

$\therefore y = 2\tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right] = 2\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right] \qquad \Rightarrow y = 2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$

$\Rightarrow y = 2\tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right] = 2\tan^{-1}\left[\tan\frac{\theta}{2}\right] = 2\times\frac{\theta}{2} = \theta$

By using (i), we get : $y = \tan^{-1}x = LHS$. Hence proved.

Q19. Let $y = \tan^{-1}\alpha + \cot^{-1}(\alpha+1) \qquad \Rightarrow y = \tan^{-1}\alpha + \tan^{-1}\left(\frac{1}{\alpha+1}\right)$

$\Rightarrow y = \tan^{-1}\left(\frac{\alpha+\frac{1}{\alpha+1}}{1-\alpha\times\frac{1}{\alpha+1}}\right) \qquad \Rightarrow y = \tan^{-1}\left(\frac{\alpha^2+\alpha+1}{\alpha+1-\alpha}\right) = \tan^{-1}\left(\alpha^2+\alpha+1\right).$

Q20. **Method 1 :** Let $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right) = \theta \Rightarrow \sin 2\theta = \frac{3}{4}$.

$\Rightarrow \frac{2\tan\theta}{1+\tan^2\theta} = \frac{3}{4} \Rightarrow 3\tan^2\theta - 8\tan\theta + 3 = 0$

$\therefore \tan\theta = \frac{-(-8)\pm\sqrt{(-8)^2-4(3)(3)}}{2(3)} = \frac{8\pm2\sqrt{7}}{2(3)} = \frac{4\pm\sqrt{7}}{3}$

So, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$.

Now as we know that $-\frac{\pi}{2} \le \sin^{-1}\frac{3}{4} \le \frac{\pi}{2} \qquad \Rightarrow -\frac{\pi}{4} \le \frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right) \le \frac{\pi}{4}$

$\Rightarrow \tan\left(-\frac{\pi}{4}\right) \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le \tan\frac{\pi}{4} \qquad \Rightarrow -1 \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le 1 \ldots(i)$

Therefore $\tan\theta = \frac{4+\sqrt{7}}{3}$ is ignored because by (i), the value of $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$ must be less than or equal to 1 but $\frac{4+\sqrt{7}}{3} \ge 1$.

Method 2 : Let $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right) = \theta \Rightarrow \sin 2\theta = \frac{3}{4}$

$\Rightarrow \frac{2\tan\theta}{1+\tan^2\theta} = \frac{3}{4} \Rightarrow 3\tan^2\theta - 8\tan\theta + 3 = 0$

$\therefore \tan\theta = \frac{-(-8)\pm\sqrt{(-8)^2-4(3)(3)}}{2(3)} = \frac{8\pm2\sqrt{7}}{2(3)} = \frac{4\pm\sqrt{7}}{3}$

So, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

Also $\tan\theta = \frac{4+\sqrt{7}}{3}$ is ignored because if $\tan\theta = \frac{4+\sqrt{7}}{3}$ then, $\sin 2\theta = \frac{4+\sqrt{7}}{32+8\sqrt{7}} \ne \frac{3}{4}$.

Q21. Let $y=\sin^2\left[2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right]$ $\quad$ [Put $x=\cos\theta \ldots$(i)

$\Rightarrow y=\sin^2\left[2\cot^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]=\sin^2\left[2\cot^{-1}\sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\right]$

$\Rightarrow y=\sin^2\left[2\cot^{-1}\tan\frac{\theta}{2}\right]=\sin^2\left[2\cot^{-1}\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$ $\quad \Rightarrow y=\sin^2\left[2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$

$\Rightarrow y=\sin^2(\pi-\theta)=\sin^2\theta$ $\quad \Rightarrow y=1-\cos^2\theta$ $\quad \therefore y=1-x^2$ $\quad$ By using (i)

Q22. Same as Q03 (z)

Q23. **(a)** LHS: $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)=\sin\left(\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)+\cos\left(\cos^{-1}\frac{1}{3}\right)$

$\Rightarrow \qquad =\sin\left(\tan^{-1}\frac{6}{8}\right)+\frac{1}{3}=\sin\left(\sin^{-1}\frac{6}{10}\right)+\frac{1}{3}=\frac{6}{10}+\frac{1}{3}=\frac{14}{15}=\text{RHS}$

(b) We have $2\cot^{-1}3=2\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)=\tan^{-1}\frac{3}{4}=\cot^{-1}\frac{4}{3}\ldots$(i)

Consider LHS : $\cot\left[\frac{\pi}{4}-2\cot^{-1}3\right]=\cot\left[\frac{\pi}{4}-\cot^{-1}\frac{4}{3}\right]=\frac{\cot\left(\cot^{-1}\frac{4}{3}\right)\cot\frac{\pi}{4}+1}{\cot\left(\cot^{-1}\frac{4}{3}\right)-\cot\frac{\pi}{4}}$

$\Rightarrow \qquad =\frac{\frac{4}{3}\times1+1}{\frac{4}{3}-1}=7=\text{RHS}.$

Q24. **(a)** $\cos(2\cos^{-1}x+\sin^{-1}x)=\cos(\cos^{-1}x+\sin^{-1}x+\cos^{-1}x)$

$\Rightarrow \qquad =\cos\left(\frac{\pi}{2}+\cos^{-1}x\right)=-\sin\cos^{-1}x$

Substituting x = 1/5, we get : $-\sin\cos^{-1}\frac{1}{5}=-\sin\sin^{-1}\frac{2\sqrt{6}}{5}=-\frac{2\sqrt{6}}{5}$

(b) Let $y=\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{3}{\sqrt{11}}\right)\right]\ldots$(i)

Put $\frac{1}{2}\cos^{-1}\left(\frac{3}{\sqrt{11}}\right)=\theta\ldots$(ii) $\quad \Rightarrow \cos 2\theta=\frac{3}{\sqrt{11}}$ $\quad \Rightarrow \frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{3}{\sqrt{11}}$

Applying componendo & dividendo, we get : $\frac{1-\tan^2\theta+1+\tan^2\theta}{1-\tan^2\theta-1-\tan^2\theta}=\frac{3+\sqrt{11}}{3-\sqrt{11}}$

$\Rightarrow \frac{2}{-2\tan^2\theta}=\frac{3+\sqrt{11}}{3-\sqrt{11}}$ $\quad \Rightarrow \tan^2\theta=\frac{\sqrt{11}-3}{\sqrt{11}+3}$ $\quad \Rightarrow \tan\theta=\sqrt{\frac{\sqrt{11}-3}{\sqrt{11}+3}}$

$\therefore \tan\left[\frac{1}{2}\cos^{-1}\left(\frac{3}{\sqrt{11}}\right)\right]=\sqrt{\frac{\sqrt{11}-3}{\sqrt{11}+3}}$ $\quad$ [By (i) and (ii)

Q25. **(a)** We have $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$ $\Rightarrow\left(\tan^{-1}x+\cot^{-1}x\right)^2-2\left(\tan^{-1}x\ \cot^{-1}x\right)=\frac{5\pi^2}{8}$

$\Rightarrow\left(\frac{\pi}{2}\right)^2-2\left(\tan^{-1}x\left(\frac{\pi}{2}-\tan^{-1}x\right)\right)=\frac{5\pi^2}{8}$ $\Rightarrow-2\left(\tan^{-1}x\left(\frac{\pi}{2}-\tan^{-1}x\right)\right)=\frac{5\pi^2}{8}-\frac{\pi^2}{4}=\frac{3\pi^2}{8}$

$\Rightarrow\frac{\pi}{2}\tan^{-1}x-(\tan^{-1}x)^2=-\frac{3\pi^2}{16}$ $\Rightarrow(\tan^{-1}x)^2-\frac{\pi}{2}\tan^{-1}x-\frac{3\pi^2}{16}=0$

$\Rightarrow16(\tan^{-1}x)^2-8\pi\tan^{-1}x-3\pi^2=0$ $\Rightarrow16(\tan^{-1}x)^2-12\pi\tan^{-1}x+4\pi\tan^{-1}x-3\pi^2=0$

$\Rightarrow4\tan^{-1}x\left(4\tan^{-1}x-3\pi\right)+\pi\left(4\tan^{-1}x-3\pi\right)=0$ $\Rightarrow\left(4\tan^{-1}x-3\pi\right)\left(4\tan^{-1}x+\pi\right)=0$

$\Rightarrow4\tan^{-1}x-3\pi=0$ or $4\tan^{-1}x+\pi=0$

$\therefore x=\tan\frac{3\pi}{4},\tan\left(-\frac{\pi}{4}\right)$ i.e., $x=-1,-1$. So, value of x is -1

(b) We have $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$ $\Rightarrow\cos\cos^{-1}\frac{1}{\sqrt{1+x^2}}=\sin\left(\sin^{-1}\frac{4}{5}\right)$

$\Rightarrow\frac{1}{\sqrt{1+x^2}}=\frac{4}{5}$ $\Rightarrow1+x^2=\frac{25}{16}$ $\Rightarrow x^2=\frac{9}{16}$ $\therefore x=\pm3/4$

Q26. We have $u=\cot^{-1}\sqrt{\tan\theta}-\tan^{-1}\sqrt{\tan\theta}$

Using $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ to get : $u=\frac{\pi}{2}-2\tan^{-1}\sqrt{\tan\theta}$

$\Rightarrow2\tan^{-1}\sqrt{\tan\theta}=\frac{\pi}{2}-u$ $\Rightarrow\sqrt{\tan\theta}=\tan\left(\frac{\pi}{4}-\frac{u}{2}\right)$

On squaring both the sides, we get : $\tan\theta=\tan^2\left(\frac{\pi}{4}-\frac{u}{2}\right)$ Proved.

Also $\tan^2\left(\frac{\pi}{4}-\frac{u}{2}\right)=\left(\frac{\tan\frac{\pi}{4}-\tan\frac{u}{2}}{1+\tan\frac{\pi}{4}\times\tan\frac{u}{2}}\right)^2\Rightarrow\quad=\left(\frac{1-\tan\frac{u}{2}}{1+\tan\frac{u}{2}}\right)^2$ Proved.

Hence, $\tan\theta=\tan^2\left(\frac{\pi}{4}-\frac{u}{2}\right)=\left(\frac{1-\tan\frac{u}{2}}{1+\tan\frac{u}{2}}\right)^2$.

Q27. **(a)** We have $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\alpha$ $\Rightarrow\cos^{-1}\left(\frac{x}{a}\times\frac{y}{b}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)=\alpha$

$\Rightarrow\cos\cos^{-1}\left(\frac{x}{a}\times\frac{y}{b}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)=\cos\alpha$ $\Rightarrow\frac{xy}{ab}-\frac{\sqrt{a^2-x^2}\sqrt{b^2-y^2}}{ab}=\cos\alpha$

$\Rightarrow xy-ab\cos\alpha=\sqrt{a^2-x^2}\sqrt{b^2-y^2}$ Squaring both the sides

$\Rightarrow x^2y^2-2xyab\cos\alpha+a^2b^2\cos^2\alpha=a^2b^2-a^2y^2-b^2x^2+x^2y^2$

$\Rightarrow a^2y^2+b^2x^2-2xyab\cos\alpha=a^2b^2(1-\cos^2\alpha)$ Dividing both sides by a^2b^2

$\Rightarrow\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$ Hence proved.

(b) Proceed same as in the Part (a).

Q28. $\because$ $a_1, a_2, a_3, ..., a_n$ are in AP so, $d = a_2 - a_1 = a_3 - a_2 = ... = a_n - a_{n-1}$

Now, $\tan\left[\tan^{-1}\frac{d}{1+a_1a_2} + \tan^{-1}\frac{d}{1+a_2a_3} + \tan^{-1}\frac{d}{1+a_3a_4} + ... + \tan^{-1}\frac{d}{1+a_{n-1}a_n}\right]$

$= \tan\left[\tan^{-1}\frac{a_2-a_1}{1+a_1a_2} + \tan^{-1}\frac{a_3-a_2}{1+a_2a_3} + \tan^{-1}\frac{a_4-a_3}{1+a_3a_4} + ... + \tan^{-1}\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right]$

$= \tan\left[\tan^{-1}a_2 - \tan^{-1}a_1 + \tan^{-1}a_3 - \tan^{-1}a_2 + \tan^{-1}a_4 - \tan^{-1}a_3 + ... + \tan^{-1}a_n - \tan^{-1}a_{n-1}\right]$

$= \tan\left[\tan^{-1}a_n - \tan^{-1}a_1\right] = \tan\tan^{-1}\frac{a_n-a_1}{1+a_1+a_n} = \frac{a_n-a_1}{1+a_1a_n}$

Q29. **(a)** We have $\sin^{-1}x > \cos^{-1}x$ $\quad \Rightarrow \sin^{-1}x > \frac{\pi}{2} - \sin^{-1}x$ $\quad \Rightarrow 2\sin^{-1}x > \frac{\pi}{2}$

$\Rightarrow \sin^{-1}x > \frac{\pi}{4}$ $\quad \Rightarrow \sin\sin^{-1}x > \sin\frac{\pi}{4}$ $\quad \Rightarrow x > \frac{1}{\sqrt{2}}$ $\quad$ ($\because$ Domain of $\sin^{-1}$ is $[-1,1]$)

$\therefore \frac{1}{\sqrt{2}} < x \le 1$ i.e., $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

(b) We have $\cos^{-1}x > \sin^{-1}x$ $\quad \Rightarrow \frac{\pi}{2} - \sin^{-1}x > \sin^{-1}x$ $\quad \Rightarrow \frac{\pi}{4} > \sin^{-1}x$

$\Rightarrow \sin\sin^{-1}x < \sin\frac{\pi}{4}$ $\quad \Rightarrow x < \frac{1}{\sqrt{2}}$ $\quad \therefore x \in \left[-1, \frac{1}{\sqrt{2}}\right)$.

Q30. Given $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$ $\quad \Rightarrow \frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y = \frac{2\pi}{3}$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$ ...(i) Also, $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$ ...(ii)

Adding (i) & (ii), $2\cos^{-1}x = \frac{2\pi}{3}$ $\quad \Rightarrow x = \cos\frac{\pi}{3} = \frac{1}{2}$

Also, subtracting (ii) from (i), $2\cos^{-1}y = 0$ $\quad \Rightarrow y = \cos 0 = 1$

$\therefore (x, y) = \left(\frac{1}{2}, 1\right)$.

Q31. We have $(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = (\sin^{-1}x + \cos^{-1}x)^2 - 2\sin^{-1}x\cos^{-1}x$

$\Rightarrow \quad = \frac{\pi^2}{4} - 2\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$

$\Rightarrow \quad = 2\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{8}\right]$

$\Rightarrow \quad = 2\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right]$

$\therefore$ Least value is obtained at $x = \frac{1}{\sqrt{2}}$ at which $\sin^{-1}x = \frac{\pi}{4}$ and, greatest value is obtained at $x = -1$ when $\sin^{-1}x = -\pi/2$.

So least value $= 2\left[\frac{\pi^2}{16}\right] = \frac{\pi^2}{8}$ and

Greatest value $= 2\left[\left(-\frac{\pi}{2} - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right] = \frac{5\pi^2}{4}$.

Q32. We have $\sin^{-1}6x+\sin^{-1}6\sqrt{3}x=-\frac{\pi}{2}$ $\Rightarrow \sin^{-1}6\sqrt{3}x=-\frac{\pi}{2}-\sin^{-1}6x$

$\Rightarrow \sin\sin^{-1}6\sqrt{3}x=\sin\left(-\frac{\pi}{2}-\sin^{-1}6x\right)$ $\Rightarrow 6\sqrt{3}x=-\cos\left(\sin^{-1}6x\right)$

$\Rightarrow 6\sqrt{3}x=-\sqrt{1-(\sin\sin^{-1}6x)^2}$ $\Rightarrow 108x^2=1-36x^2$ $\Rightarrow x=\pm\frac{1}{12}$

$\because x=\frac{1}{12}$ doesn't satisfy the given equation, $\therefore x=-\frac{1}{12}$ is the only required solution.

Q33. **(a)** For $\sin^{-1}2x$ to be defined, $2x\in[-1,1]$ i.e., $-1\le 2x\le 1$

$\Rightarrow -\frac{1}{2}\le x\le\frac{1}{2}$ $\therefore x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

(b) For $\sin^{-1}(-x^2)$ to be defined, $-x^2\in[-1,1]$ i.e., $-1\le -x^2\le 1$ $\Rightarrow |x|\le 1$ i.e., $x\in[-1,1]$

(c) For $\cos^{-1}(x^2-4)$ to be defined, $x^2-4\in[-1,1]$ i.e., $-1\le x^2-4\le 1$ $\Rightarrow 3\le x^2\le 5$

$\therefore -\sqrt{5}\le x\le -\sqrt{3}$ or $\sqrt{3}\le x\le\sqrt{5}$ i.e., $x\in[-\sqrt{5},-\sqrt{3}]\cup[\sqrt{3},\sqrt{5}]$

(d) For $\sin^{-1}x+\cos x$ to be defined, both the functions $\sin^{-1}x$ and $\cos x$ must be defined which is possible if $x\in[-1,1]$ and $x\in R$ respectively.

Therefore, $\sin^{-1}x+\cos x$ is defined in the interval $x\in R\cap[-1,1]$ i.e., $[-1, 1]$.

Q34. We have $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{x^2+x}=\frac{\pi}{2}-\sin^{-1}\sqrt{x^2+x+1}=\cos^{-1}\sqrt{x^2+x+1}$

$\Rightarrow \tan^{-1}\sqrt{x^2+x}=\sec^{-1}\frac{1}{\sqrt{x^2+x+1}}$

$\Rightarrow \tan^{-1}\sqrt{x^2+x}=\tan^{-1}\sqrt{\frac{-x^2-x}{x^2+x+1}}$

$\Rightarrow \tan\tan^{-1}\sqrt{x^2+x}=\tan\tan^{-1}\sqrt{\frac{-x^2-x}{x^2+x+1}}$

$\Rightarrow \sqrt{x^2+x}=\sqrt{\frac{-x^2-x}{x^2+x+1}}$

Let $\sec^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right)=\theta$

$\Rightarrow \sec\theta=\frac{1}{\sqrt{x^2+x+1}}$

$\Rightarrow \tan\theta=\sqrt{\frac{1}{x^2+x+1}-1}$

$\Rightarrow \theta=\tan^{-1}\sqrt{\frac{-x^2-x}{x^2+x+1}}$

On squaring both sides, we get

$\Rightarrow (x^2+x)(x^2+x+1)=-x^2-x$ $\Rightarrow [x^2+x][x^2+x+1+1]=0$

$\Rightarrow x^2+x=0$ or $x^2+x+2=0$ (No real solutions)

$\Rightarrow x(x+1)=0$ $\therefore x=0,\ x=-1$.

Alternatively, we have $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{x(x+1)}=\frac{\pi}{2}-\sin^{-1}\sqrt{x^2+x+1}$ $\Rightarrow \cos^{-1}\frac{1}{\sqrt{x^2+x+1}}=\cos^{-1}\sqrt{x^2+x+1}$

$\Rightarrow \cos\cos^{-1}\frac{1}{\sqrt{x^2+x+1}}=\cos\cos^{-1}\sqrt{x^2+x+1}$ $\Rightarrow \frac{1}{\sqrt{x^2+x+1}}=\sqrt{x^2+x+1}$

$\Rightarrow x^2+x+1=1$ $\Rightarrow x^2+x=0$ $\therefore x=0,\ x=-1$.

Q35. $\because 2\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\theta}{2}\right)=\cos^{-1}\left(\dfrac{a\cos\theta+b}{a+b\cos\theta}\right)$, replace $a=5,\ b=3$ and complete yourself.

Also see Q14 (a).

Q36. We have $\tan^{-1}\left(\dfrac{1}{1+1.2}\right)+\tan^{-1}\left(\dfrac{1}{1+2.3}\right)+...+\tan^{-1}\left(\dfrac{1}{1+n.(n+1)}\right)=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}\left(\dfrac{2-1}{1+1.2}\right)+\tan^{-1}\left(\dfrac{3-2}{1+2.3}\right)+...+\tan^{-1}\left(\dfrac{(n+1)-n}{1+n.(n+1)}\right)=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}2-\tan^{-1}1+\tan^{-1}3-\tan^{-1}2+...+\tan^{-1}n-\tan^{-1}(n-1)+\tan^{-1}(n+1)-\tan^{-1}n=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}(n+1)-\tan^{-1}1=\tan^{-1}\theta$ $\qquad$ $\Rightarrow \tan^{-1}\dfrac{(n+1)-1}{1+(n+1).1}=\tan^{-1}\theta$

$\Rightarrow \tan\tan^{-1}\dfrac{n}{1+n+1}=\tan\tan^{-1}\theta$ $\qquad$ $\therefore \theta=\dfrac{n}{n+2}$.