◘ Based On Algebra Of Determinants

VERY SHORT ANSWER TYPE QUESTIONS

Q01. (a) Let $\Delta = \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} = (a+ib)(a-ib)-(c+id)(-c+id)$

$\Rightarrow \quad = a^2 - i^2b^2 + c^2 - i^2d^2 = a^2+b^2+c^2+d^2$ $[\because i^2 = -1$

(b) Let $\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$\Rightarrow \Delta = \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ \omega+\omega^2+1 & \omega^2 & 1 \\ \omega^2+1+\omega & 1 & \omega \end{vmatrix}$ $\Rightarrow \Delta = \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix}$ $[\because 1+\omega+\omega^2 = 0$

Since all the elements in C_1 are 0 so, $\Delta = 0$.

(c) Let $\Delta = \begin{vmatrix} \sin 20^\circ & \cos 20^\circ \\ -\sin 70^\circ & \cos 70^\circ \end{vmatrix}$ $\Rightarrow \Delta = \sin 20^\circ \cos 70^\circ + \sin 70^\circ \cos 20^\circ$

$\Rightarrow \Delta = \sin(90^\circ - 70^\circ)\cos 70^\circ + \sin 70^\circ \cos(90^\circ - 70^\circ)$ $\Rightarrow \Delta = \cos^2 70^\circ + \sin^2 70^\circ = 1$

(d) Let $\Delta = \begin{vmatrix} p & 0 & 0 \\ a & q & 0 \\ b & c & r \end{vmatrix}$ $\Rightarrow \Delta = p(qr) - 0(ar-0) + 0(ac-bq) = pqr$

(e) Given $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\Rightarrow A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

$\Rightarrow |A^n|$ i.e., $\text{Det}(A^n) = \begin{vmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{vmatrix} = \cos^2 n\theta + \sin^2 n\theta = 1$.

(f) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ By $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$\Rightarrow \Delta = \begin{vmatrix} 0 & -\sin\theta & 0 \\ 0 & \sin\theta & -\cos\theta \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ Expanding along R_1,

$\Rightarrow \Delta = 0 + \sin\theta(0+\cos\theta) + 0 = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

Now as $-1 \le \sin 2\theta \le 1$ for all $\theta \in R$ $\Rightarrow -\frac{1}{2} \le \frac{1}{2}\sin 2\theta \le \frac{1}{2}$ for all $\theta \in R$

Clearly, maximum value of Δ is 1/2.

Q02. $\begin{vmatrix} 3x^3 & 8 \\ -4 & 4y^3 \end{vmatrix} = -4$ $\Rightarrow 12x^3y^3 + 32 = -4$ $\Rightarrow 12(xy)^3 = -36$

$\Rightarrow (xy)^3 = -3$ $\therefore xy = (-3)^{1/3}$

Q03. (a) $\begin{vmatrix} 3x & 1 \\ 5 & -x \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 7 & 2 \end{vmatrix}$ $\Rightarrow -3x^2 - 5 = -2 - 7$ $\Rightarrow -3x^2 = -9 + 5 = -4$

$\Rightarrow x^2 = \frac{4}{3}$ $\quad\quad$ $\Rightarrow x = \pm\frac{2}{\sqrt{3}}$

(b) $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$ $\quad\quad$ $\Rightarrow (x+1)(x+2)-(x-3)(x-1) = 12-(-1) = 13$

$\Rightarrow x^2+3x+2-x^2+4x-3 = 13$ $\quad\quad$ $\Rightarrow 7x-14=0$ $\quad\quad$ $\therefore x = 2$.

(c) $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$ $\quad\quad$ $\Rightarrow 2x^2 - 40 = 18+14$ $\quad$ $\Rightarrow 2x^2 = 72$

$\Rightarrow x^2 = 36$ $\quad\quad$ $\therefore x = \pm 6$.

Q04. $\begin{vmatrix} 2\sin x & -1 \\ 1 & \sin x \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ -4 & \sin x \end{vmatrix}$ $\quad\quad$ $\Rightarrow 2\sin^2 x + 1 = 3\sin x$ $\quad\quad$ $\Rightarrow 2\sin^2 x - 3\sin x + 1 = 0$

Let $\sin x = y \ldots(i)$ $\quad\quad$ $\therefore 2y^2 - 3y + 1 = 0$ $\quad\quad$ $\Rightarrow 2y^2 - 2y - y + 1 = 0$

$\Rightarrow 2y(y-1) - 1(y-1) = 0$ $\quad\quad$ $\Rightarrow (2y-1)(y-1) = 0$ $\quad\quad$ $\therefore y = \frac{1}{2}$ $\quad$ or $\quad$ $y = 1$

So by (i), $\sin x = \frac{1}{2}$ $\quad$ or $\quad$ $\sin x = 1$ $\quad\quad$ $\therefore\ x = \frac{\pi}{6}$ or, $\frac{\pi}{2}$ $\quad\quad$ $\left[\because 0 \le x \le \frac{\pi}{2}\right.$

Q05. Given $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\therefore a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = a_{21}(a_{22}a_{33} - a_{32}a_{23}) + a_{22}(-[a_{21}a_{33} - a_{31}a_{23}]) + a_{23}(a_{21}a_{32} - a_{31}a_{22})$

i.e., $\quad = a_{21}a_{22}a_{33} - a_{32}a_{23}a_{21} - a_{22}a_{21}a_{33} + a_{31}a_{23}a_{22} + a_{23}a_{21}a_{32} - a_{31}a_{22}a_{23} = 0$

Q06. Given $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad\quad$ $\Rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

On expanding along 3rd column, $|A| = a_{13}M_{13} - a_{23}M_{23} + a_{33}M_{33}$.

Q07. (a) $A = 2B$ $\quad\quad$ $\Rightarrow |A| = |2B|$ $\quad\quad$ $\Rightarrow |A| = 2^3|B|$ $\quad\quad$ $[\because |kP| = k^n|P|$, where n is order of P

$\therefore |A| = 8\times 5$ $\quad\quad$ $\Rightarrow |A| = 40$

(b) Given $A(\operatorname{adj} A) = 5I$ $\quad\quad$ $\because A(\operatorname{adj} A) = |A|I$ $\quad\quad$ $\therefore |A| = 5$

(c) As $|\operatorname{adj} A| = |A|^{n-1}$, where n is the order of matrix $\quad\quad$ $\therefore |\operatorname{adj} A| = (5)^{3-1} = 25$

(d) Using $|\operatorname{adj} A| = |A|^{n-1}$, $|\operatorname{adj} A| = |A|^{3-1} = 64 \Rightarrow |A|^2 = 64$ $\quad\quad$ $\therefore |A| = \pm 8$

(e) Using $|A^{-1}| = \frac{1}{|A|}$ $\quad\quad$ $\Rightarrow |A^{-1}| = \frac{1}{10}$ $\quad\quad$ $[\because |A| = 10$ is given

(f) We have $A(\operatorname{adj}A) = 5I$ $\quad\quad$ $\Rightarrow |A|I = 5I$ $\quad\quad$ $[\because A(\operatorname{adj}A) = |A|I$

$\therefore |A| = 5$ $\quad\quad$ So, $|\operatorname{adj} A| = |A|^{3-1} = 5^2 = 25$

(g) $\because |kA| = k^n|A|$, where n is order of A $\quad\quad$ $\therefore |kA| = k^3|A|$, as order of A is 3 here.

(h) We have $|\operatorname{adj}A| = |A|^{3-1}$ $\quad\quad$ $\Rightarrow |A|^2 = 36$ $\quad\quad$ $\Rightarrow |A| = \pm 6$ $\quad\quad$ $\Rightarrow |A^{-1}| = \frac{1}{|A|} = \pm\frac{1}{6}$

$\Rightarrow |3A^{-1}| = 3^3|A^{-1}| = 27\left(\pm\frac{1}{6}\right) = \pm\frac{9}{2}$ $\quad\quad$ $[\because |kA| = k^n|A|$, where n is order of A

(i) Using $|\operatorname{adj} A| = |A|^{n-1}$, we have $|\operatorname{adj}A| = |A|^{3-1} = |A|^2$.

(j) We have $|3A| = k|A|$ $\Rightarrow 3^3|A| = k|A|$ $\therefore k = 27$.

(k) As $|AA^T| = |A||A^T| = |A||A| = 5\times 5 = 25$.

(l) $|3AB| = 3^3|A||B| = 27\times 2\times 3 = 162$.

Q08. (a) Let $A = \begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$

Since matrix A is singular matrix so, $|A| = 0$ $\Rightarrow \begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$

$\Rightarrow 4(5-x) - 2(x+1) = 0$ $\Rightarrow 20 - 4x - 2x - 2 = 0$ $\Rightarrow 6x = 18$ $\Rightarrow x = 3$

(b) Since the given matrix is non-invertible so it must be singular i.e., $\begin{vmatrix} 2-x & 3 \\ -5 & 1 \end{vmatrix} = 0$

$\Rightarrow 2 - x + 15 = 0$ $\therefore x = 17$

(c) Since $\begin{bmatrix} 7-2x & x+5 \\ 3 & 7 \end{bmatrix}$ is singular so, $\begin{vmatrix} 7-2x & x+5 \\ 3 & 7 \end{vmatrix} = 0$

That is, $7(7-2x) - 3(x+5) = 0$ $\Rightarrow 49 - 14x - 3x - 15 = 0$ $\Rightarrow 17x = 34$ $\therefore x = 2$

(d) Let $A = \begin{bmatrix} 3 & 2\sin x \\ 2\sin x & 1 \end{bmatrix}$

Since A is singular matrix so, $|A| = 0$ $\Rightarrow \begin{vmatrix} 3 & 2\sin x \\ 2\sin x & 1 \end{vmatrix} = 0$

$\Rightarrow 3 - 4\sin^2 x = 0$ $\Rightarrow \sin^2 x = \frac{3}{4}$ $\therefore \sin x = \pm\frac{\sqrt{3}}{2}$

$\Rightarrow \sin x = \frac{\sqrt{3}}{2}$ or $\sin x = \frac{-\sqrt{3}}{2}$ [Negative value of sin x is rejected as $0 < x < \pi$

$\therefore \sin x = \frac{\sqrt{3}}{2}$ $\Rightarrow x = \frac{\pi}{3}$

(e) Do yourself. Ans. $x = 2\pi/3$

Q09. As matrix A is invertible so, it must be non-singular i.e., $|A| \neq 0$. Also since $|A| = |A^T|$ so, clearly A^T is also invertible.

Now as $AA^{-1} = I$ $\Rightarrow (AA^{-1})^T = I^T$ $\Rightarrow (A^{-1})^T A^T = I$ $\therefore (A^T)^{-1} = (A^{-1})^T$.

Q10. We have $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ $\Rightarrow |A| = -4 - 15 = -19$ and $adj.A = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

$\therefore A^{-1} = -\frac{1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$ i.e., $A^{-1} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$.

Q11. Let $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ $\Rightarrow |A| = \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix} = xyz$

Consider A_{ij} be the cofactors of corresponding element a_{ij} of matrix A,

$A_{11} = yz,$ $A_{12} = 0,$ $A_{13} = 0$

$A_{21} = 0,$ $A_{22} = xz,$ $A_{23} = 0$

$A_{31} = 0,$ $A_{32} = 0,$ $A_{33} = xy$

$$\therefore A^{-1}=\frac{1}{|A|}(\text{adj}A)=\frac{1}{xyz}\begin{bmatrix} yz & 0 & 0\\ 0 & xz & 0\\ 0 & 0 & xy\end{bmatrix} \quad \Rightarrow A^{-1}=\begin{bmatrix} x^{-1} & 0 & 0\\ 0 & y^{-1} & 0\\ 0 & 0 & z^{-1}\end{bmatrix}.$$

Q12. Since element 8 is present in second row and third column in $\Delta=\begin{vmatrix} 2 & 4 & 7\\ 3 & 6 & 8\\ -2 & -3 & 1\end{vmatrix}$

So, its minor is : $2\times(-3)-4\times(-2)=-6+8=2$.

Q13. (a) Equation of line joining the points (1, 2) and (3, 6) is given by : $\begin{vmatrix} x & y & 1\\ 1 & 2 & 1\\ 3 & 6 & 1\end{vmatrix}=0$

$\Rightarrow x(2-6)-y(1-3)+1(6-6)=0 \quad \Rightarrow -4x+2y+0=0 \quad \therefore 2x-y=0$

(b) Let $A(a,b+c)$, $B(b,c+a)$ and $C(c,a+b)$.

If A, B and C are collinear then, ar(ABC) = 0 i.e., $\Delta=\frac{1}{2}\begin{vmatrix} a & b+c & 1\\ b & c+a & 1\\ c & a+b & 1\end{vmatrix}=0$

LHS : $\Delta=\frac{1}{2}\begin{vmatrix} a & b+c & 1\\ b & c+a & 1\\ c & a+b & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix} a & a+b+c & 1\\ b & b+c+a & 1\\ c & c+a+b & 1\end{vmatrix}$ Applying $C_2\rightarrow C_2+C_1$

Since C_2 and C_3 are proportional so, $\Delta=0=$ RHS.

Therefore, the points A, B and C are collinear.

(c) Area of a triangle is 35 sq.units $=\frac{1}{2}\begin{vmatrix} x & 4 & 1\\ 2 & -6 & 1\\ 5 & 4 & 1\end{vmatrix}$

$\Rightarrow 70=|x(-6-4)-4(2-5)+1(8+30)| \quad \Rightarrow \pm 70=-10x+12+38$

$\Rightarrow 70=-10x+50$ or, $-70=-10x+50 \quad \Rightarrow 10x=-20$ or, $10x=120 \quad \therefore x=-2,12$

Q14. Let $\Delta=\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C\\ -\sin B & 0 & \tan A\\ \cos(A+B) & -\tan A & 0\end{vmatrix}$

As A, B, C are the angle of a triangle so, $A+B+C=\pi$

$$\therefore \Delta=\begin{vmatrix} \sin\pi & \sin B & \cos C\\ -\sin B & 0 & \tan A\\ \cos(\pi-c) & -\tan A & 0\end{vmatrix} \quad \Rightarrow \Delta=\begin{vmatrix} 0 & \sin B & \cos C\\ -\sin B & 0 & \tan A\\ -\cos C & -\tan A & 0\end{vmatrix}$$

$\Rightarrow \Delta=0-\sin B(0+\cos C\tan A)+\cos C(\sin B\tan A-0)$

$\Rightarrow \Delta=-\sin B\tan A\cos C+\sin B\tan A\cos C=0$

Q15. Since $A(\text{adj}A)=|A|I$ so, $A(\text{adj}A)=[1(3-0)-2(9-2)+3(0-1)]I$

$\Rightarrow A(\text{adj}A)=-14I_3 \quad \left[\because |A|=\begin{vmatrix} 1 & 2 & 3\\ 3 & 1 & 2\\ 1 & 0 & 3\end{vmatrix}=-14.\right.$

Q16. (a) Let $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$ $\quad \because R_1$ and R_3 are identical $\quad \therefore \Delta = 0$

(b) Let $\Delta = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$ $\quad$ By $C_1 \to C_1 + 9C_2$

$\Rightarrow \Delta = \begin{vmatrix} 65 & 7 & 65 \\ 75 & 8 & 75 \\ 86 & 9 & 86 \end{vmatrix}$ $\quad \because C_1$ and C_2 are identical $\quad \therefore \Delta = 0$

(c) Let $\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$ $\quad \because R_1$ and R_3 are proportional so, $\Delta = 0$

(d) Let $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ $\quad$ By $C_3 \to C_3 + C_2$

$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$ $\quad \because C_1$ and C_3 are proportional $\quad \therefore \Delta = 0$

(e) Let $\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$ $\quad$ By $R_1 \to R_1 + R_2 + R_3$

$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & 0 \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$ $\quad \because$ all the elements in R_1 are zero $\quad \therefore \Delta = 0$

(f) Let $\Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$ $\quad$ By $R_2 \to R_2 - R_1$

$\Rightarrow \Delta = \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$ $\quad \because R_2$ and R_3 are proportional $\quad \therefore \Delta = 0$

(g) Let $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ $\quad$ By $R_1 \to R_1 + R_2$

$\Rightarrow \Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ $\quad \because R_1$ and R_3 are proportional $\quad \therefore \Delta = 0$

(h) Let $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$\Rightarrow \Delta = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$ $\because$ all the elements in C_1 are zero $\therefore \Delta = 0$

(i) Let $\Delta = \begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ab \\ 1 & ab & ca+bc \end{vmatrix}$ By $C_3 \to C_3 + C_2$

$\Rightarrow \Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$ $\because C_3$ and C_1 are proportional $\therefore \Delta = 0$.

(j) Let $A = \begin{pmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{pmatrix}$ $\Rightarrow A^T = \begin{pmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{pmatrix} = -A$

As $|A^T| = |A|$ so, $|-A| = |A|$ $\Rightarrow (-1)^3 |A| = |A|$ $\Rightarrow -|A| = |A|$ $\Rightarrow 2|A| = 0$

Therefore, $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix} = 0$.

Q17. (a) Let $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$\Rightarrow \Delta = \begin{vmatrix} 0 & a-b & c(b-a) \\ 0 & b-c & a(c-b) \\ 1 & c & ab \end{vmatrix}$ Taking $(a-b)$ & $(b-c)$ common from R_1 & R_2 respectively

$\Rightarrow \Delta = (a-b)(b-c) \begin{vmatrix} 0 & 1 & -c \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$ By $R_1 \to R_1 - R_2$

$\Rightarrow \Delta = (a-b)(b-c) \begin{vmatrix} 0 & 0 & a-c \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$ Taking $(c-a)$ common from R_1

$\Rightarrow \Delta = (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$ Now expanding along R_1

$\Rightarrow \Delta = (a-b)(b-c)(c-a)\{0-0-1(0-1)\}$ $\therefore \Delta = (a-b)(b-c)(c-a)$

(b) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$ By $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ z(y-x) & x(z-y) & xy \end{vmatrix}$$ Taking $x-y$ & $y-z$ common from C_1 & C_2 respectively

$$\Rightarrow \Delta = (x-y)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ -z & -x & xy \end{vmatrix}$$ By $C_1 \to C_1 - C_2$

$$\Rightarrow \Delta = (x-y)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & z \\ x-z & -x & xy \end{vmatrix}$$ Taking $(z-x)$ common from C_1

$$\Rightarrow \Delta = (x-y)(y-z)(z-x)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & z \\ -1 & -x & xy \end{vmatrix}$$ Expanding along C_1

$$\Rightarrow \Delta = (x-y)(y-z)(z-x)\{0-0-1(0-1)\}$$ $\therefore \Delta = (x-y)(y-z)(z-x)$

(c) Let $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$ Taking $2(x+y)$ common from R_1

$$\Rightarrow \Delta = 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$ By $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$\Rightarrow \Delta = 2(x+y)\begin{vmatrix} 0 & 0 & 1 \\ -x & y & x \\ y & x-y & y \end{vmatrix}$$ Expanding along R_1

$$\Rightarrow \Delta = 2(x+y)\{0-0+1(-x^2+xy-y^2)\}$$

$$\Rightarrow \Delta = -2(x+y)(x^2-xy+y^2)$$ $[\because a^3+b^3=(a+b)(a^2-ab+b^2)$

$$\therefore \Delta = -2(x^3+y^3)$$

(d) Let $\Delta = \begin{vmatrix} x+a & a & a \\ b & x+b & b \\ c & c & x+c \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} x+a+b+c & x+a+b+c & x+a+b+c \\ b & x+b & b \\ c & c & x+c \end{vmatrix}$$ Taking $(x+a+b+c)$ common from R_1

$$\Rightarrow \Delta = (x+a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & x+b & b \\ c & c & x+c \end{vmatrix}$$ By $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$\Rightarrow \Delta = (x+a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ -x & x & b \\ 0 & -x & x+c \end{vmatrix}$$ Taking x common from C_1 & C_2 both

$$\Rightarrow \Delta = x^2(x+a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & b \\ 0 & -1 & x+c \end{vmatrix}$$ Expanding along R_1

$$\Rightarrow \Delta = x^2(x+a+b+c)\{0-0+1(1-0)\}$$ $\therefore \Delta = x^2(x+a+b+c)$

(e) Let $\Delta = \begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 2(y+z) & 2(x+z) & 2(x+y) \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$$ Taking 2 common from R_1

$$\Rightarrow \Delta = 2\begin{vmatrix} y+z & x+z & x+y \\ y & x+z & y \\ z & z & x+y \end{vmatrix}$$ By $R_1 \to R_1 - (R_2 + R_3)$

$$\Rightarrow \Delta = 2\begin{vmatrix} 0 & -z & -y \\ y & x+z & y \\ z & z & x+y \end{vmatrix}$$ By $R_2 \to R_2 + R_1, R_3 \to R_3 + R_1$

$$\Rightarrow \Delta = 2\begin{vmatrix} 0 & -z & -y \\ y & x & 0 \\ z & 0 & x \end{vmatrix}$$ Now expanding along R_1

$$\Rightarrow \Delta = 2[0 + z(xy-0) - y(0-xz)] = 4xyz$$

(f) Let $\Delta = \begin{vmatrix} 0 & -b & c \\ b & 0 & a \\ -c & -a & 0 \end{vmatrix}$ By $R_1 \to aR_1, R_2 \to cR_2, R_3 \to bR_3$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} 0 & -ab & ca \\ bc & 0 & ac \\ -bc & -ab & 0 \end{vmatrix}$$ Taking bc, ab & ca common from C_1, C_2 & C_3 resp.

$$\Rightarrow \Delta = \frac{bc \times ab \times ac}{abc}\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix}$$ Expanding along R_1

$$\Rightarrow \Delta = abc[0 + 1(0+1) + 1(-1+0)] = 0$$

(g) Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$ By $R_1 \to cR_1$

$$\Rightarrow \Delta = \frac{1}{c}\begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$ By $R_1 \to R_1 - bR_2$

$\Rightarrow \Delta = \frac{1}{c}\begin{vmatrix} ab & ac & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$ Taking a common from R_1,

$\Rightarrow \Delta = \frac{a}{c}\begin{vmatrix} b & c & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$ Since R_1 & R_3 are same, $\therefore \Delta = 0$

(h) Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$ Expanding along C_3,

$\Rightarrow \Delta = -\sin\alpha(-\sin\alpha\sin^2\beta - \cos^2\beta\sin\alpha) - 0 + \cos\alpha(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta)$

$\Rightarrow \Delta = \sin^2\alpha(\sin^2\beta + \cos^2\beta) + \cos^2\alpha(\cos^2\beta + \sin^2\beta) = 1$

(i) Let $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$ By $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$\Rightarrow \Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix} = 1\left[1(xy-0) - 0 + 0\right] = xy$ [We've expanded Δ along C_1.

(j) Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$ By $C_1 \to \sin\delta C_1,\ C_2 \to \cos\delta C_2$

$\Rightarrow \Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos(\gamma+\delta) \end{vmatrix}$ By $C_1 \to C_1 + C_3$

$\Rightarrow \Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \cos\alpha\cos\delta & \cos\alpha\cos\delta & \cos(\alpha+\delta) \\ \cos\beta\cos\delta & \cos\beta\cos\delta & \cos(\beta+\delta) \\ \cos\gamma\cos\delta & \cos\gamma\cos\delta & \cos(\gamma+\delta) \end{vmatrix}$ [Using $\cos(A+B) = \cos A\cos B - \sin A\sin B$ in C_3

Since C_1 and C_2 are identical so, $\Delta = \frac{1}{\sin\delta\cos\delta} \times 0 = 0$.

Q18. Consider $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix} \Rightarrow \Delta_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix}$ (Interchanging rows and columns)

$\Rightarrow \Delta_1 = \frac{1}{xyz}\begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$ (By $R_1 \to xR_1,\ R_2 \to yR_2,\ R_3 \to zR_3$)

$\Rightarrow \Delta_1 = \frac{xyz}{xyz}\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}$ (Taking xyz common from C_2)

$$\Rightarrow \Delta_1 = -\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad \text{(Interchanging } C_1 \text{ and } C_2\text{)}$$

$$\Rightarrow \Delta_1 = -\Delta \qquad \therefore \Delta + \Delta_1 = 0.$$

SHORT & LONG ANSWER TYPE QUESTIONS – TYPE A

Q01. (a) Method 1 : We have $AB - CD = O$ $\Rightarrow AB = CD$ Pre-multiplying by C^{-1} both sides,

$\Rightarrow C^{-1}AB = C^{-1}CD$ $\Rightarrow C^{-1}AB = ID$ $\Rightarrow D = C^{-1}AB$

$$\Rightarrow D = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} \qquad \text{Using } P^{-1} = \frac{1}{|P|}\text{adj}P \text{ to obtain } C^{-1}$$

$$\Rightarrow D = \frac{1}{4-2}\begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 6 \end{bmatrix} \qquad \Rightarrow D = -\frac{1}{2}\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 6 \end{bmatrix}$$

$$\Rightarrow D = -\frac{1}{2}\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} \qquad \therefore D = \begin{bmatrix} 2 & 0 \\ 0 & -3/2 \end{bmatrix}$$

Method 2 : Let $D = \begin{bmatrix} u & v \\ x & y \end{bmatrix}$ Now, $AB - CD = O$ $\Rightarrow AB = CD$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} u & v \\ x & y \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} -2 & -3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 2x-u & 2y-v \\ u-4x & v-4y \end{bmatrix}$$

By equality of matrices, we have : $2x - u = -2,\ 2y - v = -3,\ u - 4x = 2,\ v - 4y = 6$

On solving these equations simultaneously, we get : $u = 2,\ v = 0,\ x = 0,\ y = -3/2$

Therefore, matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & -3/2 \end{bmatrix}$.

(b) Do yourself. Ans. $D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$

Q02. We have $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = I$ $\Rightarrow PAQ = I$, where $P = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$

$\Rightarrow P^{-1}PAQ = P^{-1}I$ [Pre-multiplying by P^{-1} both sides

$\Rightarrow IAQQ^{-1} = P^{-1}Q^{-1}$ [Post-multiplying by Q^{-1} both sides

$$\Rightarrow AI = P^{-1}Q^{-1} \qquad \Rightarrow A = P^{-1}Q^{-1} \qquad \Rightarrow A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1}\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}^{-1}$$

$$\Rightarrow A = \left(\frac{1}{4-3}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\right)\left(\frac{1}{9-10}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}\right) \qquad \Rightarrow A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 6-5 & 4-3 \\ -9+10 & -6+6 \end{bmatrix} \qquad \therefore A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Q03. **(a)** We have $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ $\Rightarrow AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$$\text{LHS : } (AB)^{-1} = \frac{1}{14-25}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} \ldots\text{(i)}$$

RHS : $B^{-1}A^{-1} = \frac{1}{3-2}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \times \frac{1}{-8-3}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} \ldots$(ii)

By (i) & (ii), we get : $(AB)^{-1} = B^{-1}A^{-1}$ i.e., LHS = RHS.

(b) Do yourself.

Q04. Given $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and, $B^{-1} = \begin{bmatrix} -1 & 0 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow \text{Adj}A = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$ & $|A| = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 4\times 2 - 3\times 1 = 5$ $\quad\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$

Now let $P = (AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -1 & 0 \\ 3 & 4 \end{bmatrix} \times \frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & 1 \\ 0 & 5 \end{bmatrix}$

Q05. **(a)** We've $A = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$ $\quad \Rightarrow \text{adj}.A = \begin{bmatrix} 4 & -3 \\ -1 & -2 \end{bmatrix}^T = \begin{bmatrix} 4 & -1 \\ -3 & -2 \end{bmatrix}$

$\therefore A(\text{adj}A) = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} = -11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ldots$(i)

and $(\text{adj}A)A = \begin{bmatrix} 4 & -1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} = -11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ldots$(ii)

Also, $|A| = \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = -8 - 3 = -11$ $\quad \Rightarrow |A|\, I = -11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -11\, I$...(iii)

By (i), (ii) & (iii), $A(\text{adj}A) = (\text{adj}A)A = |A|\, I$ is verified.

(b) Given $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Consider A_{ij} be the cofactors of corresponding elements a_{ij} of matrix A then,

$A_{11} = 7,\quad A_{12} = -1,\quad A_{13} = -1$

$A_{21} = -3,\quad A_{22} = 1,\quad A_{23} = 0$

$A_{31} = -3,\quad A_{32} = 0,\quad A_{33} = 1$

$\therefore \text{adj}A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Also, $A(\text{adj}A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$...(i)

And, $|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1(16-9) - 3(4-3) + 3(3-4) = 7 - 3 - 3 = 1$

So, $|A|\, I = 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$...(ii)

By (i) and (ii), we get : $A(\text{adj}A) = |A|\, I$. Hence Verified.

Also, $A^{-1} = \dfrac{\text{adj}A}{|A|} = \dfrac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

(c) Given $A = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ $\Rightarrow |A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} = 27$

Consider A_{ij} be the cofactors of the corresponding element a_{ij} of matrix A.

$A_{11} = -3,\ A_{12} = -6,\ A_{13} = -6;$

$A_{21} = 6,\ A_{22} = 3,\ A_{23} = -6;$ $\Rightarrow \text{adj.}A = \begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix}$

$A_{31} = 6,\ A_{32} = -6,\ A_{33} = 3.$

LHS: $A.(\text{adj}.A) = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix} = \begin{pmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix} = 27\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = |A|\, I_3 =$ RHS

Q06. LHS : $\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1}$

$$\Rightarrow \quad = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\frac{1}{\left[1+\tan^2\frac{\theta}{2}\right]}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\frac{1}{\sec^2\frac{\theta}{2}}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} \cos^2\frac{\theta}{2} & -\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} 1 & -\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\ \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} & 1 \end{bmatrix}\begin{bmatrix} \cos^2\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) & \cos^2\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2} & -\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2} \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} \cos\theta & -\sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right) \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \Rightarrow \quad = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \text{RHS}.$$

Q07. We have $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$$\Rightarrow \quad = 1\left(1+\sin^2\theta\right) - \sin\theta\left(-\sin\theta+\sin\theta\right) + 1\left(\sin^2\theta+1\right) = 2(1+\sin^2\theta)$$

That is, $\Delta = 2 + 2\sin^2\theta$

$\because -1 \le \sin\theta \le 1 \qquad \Rightarrow 0 \le \sin^2\theta \le 1 \qquad \Rightarrow 0 \le 2\sin^2\theta \le 2$

$\Rightarrow 2+0 \le 2+2\sin^2\theta \le 2+2 \qquad \Rightarrow 2 \le \Delta \le 4$. Hence Proved.

Note that in Q07, there is a MCQ optional question as well, which can be solved exactly in the same way as given above and Option (b) will be correct for this sum.

Q08. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix} \qquad \Rightarrow A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} \qquad \therefore |A'| = 1(-9) - 0 - 2(-5) = 1$

Consider A_{ij} as the cofactor of corresponding element a_{ij} of matrix A'.

$A_{11} = -9,\ A_{12} = 8,\ A_{13} = -5$,

$A_{21} = -8,\ A_{22} = 7,\ A_{23} = -4$,

$A_{31} = -2,\ A_{32} = 2,\ A_{33} = -1$

$\therefore\ \text{adj}(A') = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

$$\text{So}, (A')^{-1} = \frac{\text{adj}(A')}{|A'|} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

Note that we can alternatively use the property, $(A')^{-1} = (A^{-1})'$.

Q09. a) $A\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ Post-multiplying by $\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$ both sides,

$$\Rightarrow A\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}^{-1} \qquad \Rightarrow AI = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}\left(\frac{1}{6+4}\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}\right)$$

$$\Rightarrow A = \frac{1}{10}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \qquad \Rightarrow A = \frac{1}{10}\begin{bmatrix} 20 & 10 \\ 10 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

b) Given $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = 6\ I_2 \qquad \Rightarrow A = 6I\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \qquad \Rightarrow A = 6I\left(\frac{1}{4+2}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\right)$

$$\Rightarrow A = 6 \times \frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \qquad \therefore A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

c) Given $\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} A = \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix} \qquad \Rightarrow A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$

$$\Rightarrow A = \left(\frac{1}{5-8}\begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix}\right)\begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix} \qquad \Rightarrow A = \left(\frac{1}{-3}\begin{bmatrix} 5 & -4 \\ -2 & 1 \end{bmatrix}\right)\begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$$

$\Rightarrow A=\frac{1}{3}\begin{bmatrix}-5&4\\2&-1\end{bmatrix}\begin{bmatrix}-1&2\\0&4\end{bmatrix}$ $\Rightarrow A=\frac{1}{3}\begin{bmatrix}5&6\\-2&0\end{bmatrix}$ $\therefore A=\begin{bmatrix}5/3&2\\-2/3&0\end{bmatrix}$

d) Given $\begin{bmatrix}1&-4\\3&-2\end{bmatrix}A=\begin{bmatrix}-16&-6\\7&2\end{bmatrix}$ $\Rightarrow A=\begin{bmatrix}1&-4\\3&-2\end{bmatrix}^{-1}\begin{bmatrix}-16&-6\\7&2\end{bmatrix}$

$\Rightarrow A=\left(\frac{1}{-2+12}\begin{bmatrix}-2&4\\-3&1\end{bmatrix}\right)\begin{bmatrix}-16&-6\\7&2\end{bmatrix}$ $\Rightarrow A=\frac{1}{10}\begin{bmatrix}-2&4\\-3&1\end{bmatrix}\begin{bmatrix}-16&-6\\7&2\end{bmatrix}$

$\Rightarrow A=\frac{1}{10}\begin{bmatrix}60&20\\55&20\end{bmatrix}$ $\therefore A=\begin{bmatrix}6&2\\11/2&2\end{bmatrix}$

e) Given $\begin{bmatrix}2&1\\3&2\end{bmatrix}A\begin{bmatrix}-3&2\\5&-3\end{bmatrix}=\begin{bmatrix}-2&4\\3&-1\end{bmatrix}$ $\Rightarrow A=\begin{bmatrix}2&1\\3&2\end{bmatrix}^{-1}\begin{bmatrix}-2&4\\3&-1\end{bmatrix}\begin{bmatrix}-3&2\\5&-3\end{bmatrix}^{-1}$

$\Rightarrow A=\left(\frac{1}{4-3}\begin{bmatrix}2&-1\\-3&2\end{bmatrix}\right)\begin{bmatrix}-2&4\\3&-1\end{bmatrix}\left(\frac{1}{9-10}\begin{bmatrix}-3&-2\\-5&-3\end{bmatrix}\right)$

$\Rightarrow A=\begin{bmatrix}2&-1\\-3&2\end{bmatrix}\begin{bmatrix}-2&4\\3&-1\end{bmatrix}\begin{bmatrix}3&2\\5&3\end{bmatrix}$ $\Rightarrow A=\begin{bmatrix}2&-1\\-3&2\end{bmatrix}\begin{bmatrix}14&8\\4&3\end{bmatrix}$ $\therefore A=\begin{bmatrix}24&13\\-34&-18\end{bmatrix}$

Q10. We have $A=\begin{pmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{pmatrix}$

$\therefore |A|=\begin{vmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{vmatrix}=3(12-10)+1(-30+25)+1(30-30)=1\neq 0\ \therefore A^{-1}$ exists.

Consider A_{ij} be the cofactors of the corresponding element a_{ij} of matrix A.

$A_{11}=2,\quad A_{12}=5\quad A_{13}=0,$

$A_{21}=0,\quad A_{22}=1,\quad A_{23}=1,$

$A_{31}=-1,\quad A_{32}=0,\quad A_{33}=3$

So, adj.A $=\begin{pmatrix}2&0&-1\\5&1&0\\0&1&3\end{pmatrix}$ $\Rightarrow A^{-1}=\frac{\text{adj.}A}{|A|}=\begin{pmatrix}2&0&-1\\5&1&0\\0&1&3\end{pmatrix}$

$\therefore A^{-1}A=\begin{pmatrix}2&0&-1\\5&1&0\\0&1&3\end{pmatrix}\begin{pmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{pmatrix}=\begin{pmatrix}6+0-5&-2+0+2&2+0-2\\15-15+0&-5+6+0&5-5+0\\0-15+15&0+6-6&0-5+6\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I$

Hence $A^{-1}.A=I$.

Q11. Here $A=\begin{pmatrix}\cos\alpha&-\sin\alpha&0\\\sin\alpha&\cos\alpha&0\\0&0&1\end{pmatrix}$ $\Rightarrow |A|=\cos\alpha(\cos\alpha-0)+\sin\alpha(\sin\alpha-0)+0=1$.

Consider A_{ij} be the cofactor of the element $a_{ij}A_{ij}$,

$A_{11}=\cos\alpha,\quad A_{12}=-\sin\alpha,\quad A_{13}=0,$

$A_{21}=\sin\alpha,\quad A_{22}=\cos\alpha,\quad A_{23}=0,$

$A_{31}=0,\quad A_{32}=0,\quad A_{33}=1$

$\therefore \text{adj.}A=\begin{pmatrix}\cos\alpha&\sin\alpha&0\\-\sin\alpha&\cos\alpha&0\\0&0&1\end{pmatrix}$

$$\text{Now } A(\text{adj}.A)=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=I=|A|I\ldots(i)$$

$$(\text{adj}.A)A=\begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=I=|A|I\ldots(ii)$$

By (i) and (ii), it is clear that $A(\text{adj}.A)=(\text{adj}.A)A=|A|I_3$.

SHORT & LONG ANSWER TYPE QUESTIONS – TYPE B

Q01. Let $\Delta=\begin{vmatrix}x & \sin\delta & \cos\delta\\ -\sin\delta & -x & 1\\ \cos\delta & 1 & x\end{vmatrix}$ Expanding along R_1

$\Delta=x\left(-x^2-1\right)-\sin\delta\left(-x\sin\delta-\cos\delta\right)+\cos\delta\left(-\sin\delta+x\cos\delta\right)$

$\Rightarrow \quad =-x^3-x+x\sin^2\delta+\sin\delta\cos\delta-\sin\delta\cos\delta+x\cos^2\delta$

$\Rightarrow \quad =-x^3-x+x\left(\sin^2\delta+\cos^2\delta\right)=-x^3-x+x\left(1\right)$

$\Rightarrow \Delta=-x^3 \quad \therefore \Delta$ is independent of δ.

Or, Given that $\begin{vmatrix}x & \sin\theta & \cos\theta\\ -\sin\theta & -x & 1\\ \cos\theta & 1 & x\end{vmatrix}=8$

$\Rightarrow x(-x^2-1)-\sin\theta(-x\sin\theta-\cos\theta)+\cos\theta(-\sin\theta+x\cos\theta)=8$

$\Rightarrow -x^3-x+x\sin^2\theta+\sin\theta\cos\theta-\cos\theta\sin\theta+x\cos^2\theta=8=\Rightarrow -x^3-x+x(\sin^2\theta+\cos^2\theta)=8$

$\Rightarrow -x^3=8 \quad \therefore x=-2$.

Q02. LHS : Let $\Delta=\begin{vmatrix}a^2 & a^2-(b-c)^2 & bc\\ b^2 & b^2-(c-a)^2 & ca\\ c^2 & c^2-(a-b)^2 & ab\end{vmatrix}$ By $C_1\to C_1-C_2$

$\Rightarrow \Delta=\begin{vmatrix}(b-c)^2 & a^2-(b-c)^2 & bc\\ (c-a)^2 & b^2-(c-a)^2 & ca\\ (a-b)^2 & c^2-(a-b)^2 & ab\end{vmatrix}$ By $C_2\to C_2+C_1$

$\Rightarrow \Delta=\begin{vmatrix}(b-c)^2 & a^2 & bc\\ (c-a)^2 & b^2 & ca\\ (a-b)^2 & c^2 & ab\end{vmatrix}$ By $C_1\to C_1+2C_3$

$\Rightarrow \Delta=\begin{vmatrix}b^2+c^2 & a^2 & bc\\ c^2+a^2 & b^2 & ca\\ a^2+b^2 & c^2 & ab\end{vmatrix}$ By $C_1\to C_1+C_2$

$\Rightarrow \Delta=\begin{vmatrix}a^2+b^2+c^2 & a^2 & bc\\ b^2+c^2+a^2 & b^2 & ca\\ a^2+b^2+c^2 & c^2 & ab\end{vmatrix}$ Taking $(a^2+b^2+c^2)$ common from C_1,

$$\Rightarrow \Delta = (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$ By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = (a^2+b^2+c^2)\begin{vmatrix} 0 & a^2-b^2 & c(b-a) \\ 0 & b^2-c^2 & a(c-b) \\ 1 & c^2 & ab \end{vmatrix}$$

Taking (a – b) and (b – c) common from R_1 and R_2 respectively.

$$\Rightarrow \Delta = (a-b)(b-c)(a^2+b^2+c^2)\begin{vmatrix} 0 & a+b & -c \\ 0 & b+c & -a \\ 1 & c^2 & ab \end{vmatrix}$$ By $R_2 \to R_2 - R_1$

$$\Rightarrow \Delta = (a-b)(b-c)(a^2+b^2+c^2)\begin{vmatrix} 0 & a+b & -c \\ 0 & c-a & c-a \\ 1 & c^2 & ab \end{vmatrix}$$ Taking (c – a) common from R_2,

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)(a^2+b^2+c^2)\begin{vmatrix} 0 & a+b & -c \\ 0 & 1 & 1 \\ 1 & c^2 & ab \end{vmatrix}$$ Expanding along C_1,

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)(a^2+b^2+c^2)\left[0-0+1(a+b+c)\right]$$

$$\therefore \Delta = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2) = \text{RHS}.$$

Or, LHS : $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$ By $C_1 \to C_1 + C_2 - 2C_3$

$$\Rightarrow \Delta = \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix}$$ Taking $a^2+b^2+c^2$ common from C_1,

$$\Rightarrow \Delta = (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$ By $R_1 \Rightarrow R_1 - R_2,\ R_2 \Rightarrow R_2 - R_3$

$$\Rightarrow \Delta = (a^2+b^2+c^2)\begin{vmatrix} 0 & a^2-b^2 & bc-ca \\ 0 & b^2-c^2 & ca-ab \\ 1 & c^2 & ab \end{vmatrix}$$ Taking $(a-b), (b-c)$ common from R_1 and R_2,

$$\Rightarrow \Delta = (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} 0 & a+b & -c \\ 0 & b+c & -a \\ 1 & c^2 & ab \end{vmatrix}$$ By $R_2 \Rightarrow R_2 - R_1$

$$\Rightarrow \Delta = (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} 0 & a+b & -c \\ 0 & c-a & c-a \\ 1 & c^2 & ab \end{vmatrix}$$ Taking (c – a) common from R_2,

$$\Rightarrow \Delta = (a^2+b^2+c^2)(a-b)(b-c)(c-a)\begin{vmatrix} 0 & a+b & -c \\ 0 & 1 & 1 \\ 1 & c^2 & ab \end{vmatrix}$$

Expanding along C_1, we have : $\Delta = (a^2+b^2+c^2)(a-b)(b-c)(c-a)\left[0-0+1\left(a+b+c\right)\right]$

$\therefore \Delta = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2) = \text{RHS}$.

Q03. (a) LHS : Let $\Delta = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$ By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & a-b & a^2-b^2-bc+ac \\ 0 & b-c & b^2-c^2-ac+ab \\ 1 & c & c^2-ab \end{vmatrix} \quad \text{or, } \Delta = \begin{vmatrix} 0 & a-b & (a-b)(a+b)+c(a-b) \\ 0 & b-c & (b-c)(b+c)+a(b-c) \\ 1 & c & c^2-ab \end{vmatrix}$$

Taking $(a-b)$ & $(b-c)$ common from R_1 & R_2 respectively

$$\Rightarrow \Delta = (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & b+c+a \\ 1 & c & c^2-ab \end{vmatrix} \quad \text{By } R_1 \to R_1 - R_2$$

$$\Rightarrow \Delta = (a-b)(b-c)\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & b+c+a \\ 1 & c & c^2-ab \end{vmatrix} \quad \text{Since all the elements in } R_1 \text{ are } 0$$

$\therefore \Delta = 0 = \text{RHS}$

(b) LHS : Let $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$ By $R_1 \to aR_1,\ R_2 \to bR_2,\ R_3 \to cR_3$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ac \\ ca^2b^2 & abc & ac+bc \end{vmatrix} \quad \text{Taking } abc \text{ common from } C_1 \text{ \& } C_2$$

$$\Rightarrow \Delta = \frac{abc \times abc}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix} \quad \text{By } C_3 \to C_3 + C_1$$

$$\Rightarrow \Delta = abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix} \quad \because C_2 \text{ and } C_3 \text{ are proportional}$$

$\therefore \Delta = 0 = \text{RHS}.$

(c) LHS : Let $\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$ By $R_1 \to aR_1,\ R_2 \to bR_2,\ R_3 \to cR_3$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ ac^2 & bc^2 & c(c^2+1) \end{vmatrix}$$

Taking a, b, c common from C_1, C_2 & C_3 respectively

$$\Rightarrow \Delta = \frac{abc}{abc}\begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \quad \text{By } R_1 \to R_1+R_2+R_3$$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Taking $(1+a^2+b^2+c^2)$ common from R_1

$$\Rightarrow \Delta = (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \quad \text{By } C_1 \to C_1 - C_2 \text{ \& } C_2 \to C_2 - C_3$$

$$\Rightarrow \Delta = (1+a^2+b^2+c^3)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & b^2 \\ 0 & -1 & c^2+1 \end{vmatrix} \quad \text{Now expanding along } R_1$$

$$\Rightarrow \Delta = (1+a^2+b^2+c^2)\{0-0+1(1-0)\} \qquad \therefore \Delta = (1+a^2+b^2+c^2) = \text{RHS}$$

(d) LHS : Let $\Delta = \begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$ By $R_1 \to R_1+R_2+R_3$

$$\Rightarrow \Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} \quad \text{Taking } (x+y+z) \text{ common from } R_1$$

$$\Rightarrow \quad = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} \quad \text{By } C_1 \to C_1 - C_2 \text{ \& } C_1 \to C_2 - C_3$$

$$= (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ x+y+z & -(x+y+z) & 2y \\ 0 & x+y+z & z-x-y \end{vmatrix} \quad \text{Taking } x+y+z \text{ common from } C_1 \text{ and } C_2 \text{ both}$$

$$\Rightarrow \quad = (x+y+z)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2y \\ 0 & 1 & z-x-y \end{vmatrix} \quad \text{Expanding along } R_1$$

$$\Rightarrow \Delta = (x+y+z)^3\{0-0+1(1+0)\} \qquad \therefore \Delta = (x+y+z)^3 = \text{RHS}$$

(e) LHS : Let $\Delta = \begin{vmatrix} 1 & \alpha & \alpha^2+\beta\gamma \\ 1 & \beta & \beta^2+\alpha\gamma \\ 1 & \gamma & \gamma^2+\alpha\beta \end{vmatrix}$ By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & \alpha-\beta & \alpha^2-\beta^2+\beta\gamma-\alpha\gamma \\ 0 & \beta-\gamma & \beta^2-\gamma^2+\alpha\gamma-\alpha\beta \\ 1 & \gamma & \gamma^2+\alpha\beta \end{vmatrix} \quad \text{or, } \Delta = \begin{vmatrix} 0 & \alpha-\beta & (\alpha-\beta)(\alpha+\beta)-\gamma(\alpha-\beta) \\ 0 & \beta-\gamma & (\beta-\gamma)(\beta+\gamma)-\alpha(\beta-\gamma) \\ 1 & \gamma & \gamma^2+\alpha\beta \end{vmatrix}$$

Taking $(\alpha-\beta)$ & $(\beta-\gamma)$ common from R_1 and R_2 respectively,

$$\Rightarrow \Delta = (\alpha-\beta)(\beta-\gamma)\begin{vmatrix} 0 & 1 & \alpha+\beta-\gamma \\ 0 & 1 & \beta+\gamma-\alpha \\ 1 & \gamma & \gamma^2+\alpha\beta \end{vmatrix} \qquad \text{By } R_1 \to R_1 - R_2$$

$$\Rightarrow \qquad = (\alpha-\beta)(\beta-\gamma)\begin{vmatrix} 0 & 0 & 2(\alpha-\gamma) \\ 0 & 1 & \beta+\gamma-\alpha \\ 1 & \gamma & \gamma^2+\alpha\beta \end{vmatrix} \qquad \text{Taking } 2(\gamma-\alpha) \text{ common from } R_1$$

$$\Rightarrow \Delta = 2(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & \beta+\gamma-\alpha \\ 1 & 1 & \gamma^2+\alpha\beta \end{vmatrix} \qquad \text{Expanding along } R_1$$

$$\Rightarrow \qquad = 2(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\{0-0-1(0-1)\} \qquad \therefore \Delta = 2(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) = \text{RHS}$$

(f) LHS : Let $\Delta = \begin{vmatrix} x+y & z+x & y+z \\ y & z & x \\ z & x & y \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \Delta = \begin{vmatrix} 2(x+y+z) & z+x & y+z \\ x+y+z & z & x \\ x+y+z & x & y \end{vmatrix} \qquad \text{Taking } (x+y+z) \text{ common from } C_1$$

$$\Rightarrow \Delta = (x+y+z)\begin{vmatrix} 2 & z+x & y+z \\ 1 & z & x \\ 1 & x & y \end{vmatrix} \qquad \text{By } R_1 \to R_1 - (R_2 + R_3)$$

$$\Rightarrow \Delta = (x+y+z)\begin{vmatrix} 0 & 0 & (z-x) \\ 1 & z & x \\ 1 & x & y \end{vmatrix} \qquad \text{Taking } (z-x) \text{ common from } R_1$$

$$\Rightarrow \qquad = (x+y+z)(z-x)\begin{vmatrix} 0 & 0 & 1 \\ 1 & z & x \\ 1 & x & y \end{vmatrix} \qquad \text{Expanding along } R_1$$

$$\Rightarrow \Delta = (x+y+z)(z-x)\{0-0+1(x-z)\} \qquad \therefore \Delta = -(x+y+z)(z-x)^2 = \text{RHS}$$

(g) LHS : Let $\Delta = \begin{vmatrix} \sqrt{3}+\sqrt{13} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix} \Rightarrow \Delta = \begin{vmatrix} \sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15} & 5 & \sqrt{10} \\ 3 & \sqrt{15} & 5 \end{vmatrix} + \begin{vmatrix} \sqrt{13} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{26} & 5 & \sqrt{10} \\ \sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

Taking $\sqrt{3}, \sqrt{5}$ and $\sqrt{5}$ common from C_1, C_2 and C_3 respectively in det(I).
Also taking $\sqrt{13}$ common from C_1 and, $\sqrt{5}$ from C_2 and C_3 both in det(II).

$$\therefore \Delta = 5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix} + 5\sqrt{13}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{2} \\ \sqrt{5} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$\because C_1$ & C_2 are identical in det(II)

$$\Rightarrow \Delta = 5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix} + 5\sqrt{13}(0)$$

By $C_1 \to C_1 - C_2$

$$\Rightarrow \Delta = 5\sqrt{3}\begin{vmatrix} -1 & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

Expanding along C_1

$$\Rightarrow \Delta = 5\sqrt{3}\left\{-1\left(5-\sqrt{6}\right)-0+0\right\}$$

$\therefore \Delta = 5\sqrt{3}[\sqrt{6}-5] = \text{RHS}$

(h) LHS : Let $\Delta = \begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{a^2+c^2}{b} \end{vmatrix}$

By $R_1 \to cR_1$, $R_2 \to aR_2$ & $R_3 \to bR_3$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & a^2+c^2 \end{vmatrix}$$

By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} 2\left(a^2+b^2\right) & 2\left(b^2+c^2\right) & 2\left(a^2+c^2\right) \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & a^2+c^2 \end{vmatrix}$$

Taking 2 common from R_1

$$\Rightarrow \quad = \frac{2}{abc}\begin{vmatrix} a^2+b^2 & b^2+c^2 & a^2+c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & a^2+c^2 \end{vmatrix}$$

By $R_1 \to R_1 - R_2$

$$\Rightarrow \quad = \frac{2}{abc}\begin{vmatrix} b^2 & 0 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & a^2+c^2 \end{vmatrix}$$

By $R_3 \to R_3 - R_1$

$$\Rightarrow \quad = \frac{2}{abc}\begin{vmatrix} b^2 & 0 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ 0 & b^2 & a^2 \end{vmatrix}$$

By $R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = \frac{2}{abc}\begin{vmatrix} b^2 & 0 & c^2 \\ a^2 & c^2 & 0 \\ 0 & b^2 & a^2 \end{vmatrix}$$

Now, expanding along R_1

$$\Rightarrow \quad = \frac{2}{abc}\left\{b^2\left(a^2c^2-0\right)-0+c^2\left(a^2b^2-0\right)\right\}$$

$\Rightarrow \Delta = \dfrac{2}{abc}\left(2a^2b^2c^2\right) = 4abc = \text{RHS}$

(i) LHS : Let $\Delta = \begin{vmatrix} 2ab & a^2 & b^2 \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} a^2+b^2+2ab & a^2+b^2+2ab & a^2+b^2+2ab \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix}$$

Taking $\left(a^2+b^2+2ab\right)$ common from R_1,

$$\Rightarrow \quad = \left(a^2+b^2+2ab\right)\begin{vmatrix} 1 & 1 & 1 \\ a^2 & 2b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix} = (a+b)^2 \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix}$$

By $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$, $\Delta = (a+b)^2 \begin{vmatrix} 0 & 0 & 1 \\ a^2-b^2 & b^2-2ab & 2ab \\ b^2-2ab & 2ab-a^2 & a^2 \end{vmatrix}$

Now expanding along R_1, we get : $\Delta = (a+b)^2 \left\{0-0+1\left[\left(a^2-b^2\right)\left(2ab-a^2\right)-\left(b^2-2ab\right)^2\right]\right\}$

$$\Rightarrow \quad = (a+b)^2\left(2a^3b - a^4 - 2ab^3 + a^2b^2 - b^4 - 4a^2b^2 + 4ab^3\right)$$

$$\Rightarrow \quad = (a+b)^2\left(2a^3b - a^4 - b^4 - 3a^2b^2 + 2ab^3\right)$$

$$\Rightarrow \quad = -(a+b)^2\left(a^4 + a^2b^2 + b^4 - 2a^2ab - 2abb^2 + 2a^2b^2\right)$$

$$\Rightarrow \quad = -(a+b)^2\left(a^2 - ab + b^2\right)^2 = -\left[(a+b)\left(a^2-ab+b^2\right)\right]^2$$

$\therefore \Delta = -\left(a^3+b^3\right)^2 = \text{RHS}$

(j) LHS : Let $\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\Rightarrow \quad = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$ By $R_3 \to R_3 - 3R_2$

$$\Rightarrow \quad = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$ Expanding along C_1

$\Rightarrow \Delta = a(a^2-0) - 0 + 0$ $\therefore \Delta = a^3 = \text{RHS}$

(k) LHS : Let $\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$ Taking $(a+b+c)$ common from C_1

$\Rightarrow \quad = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$ By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$\Rightarrow \quad = (a+b+c)\begin{vmatrix} 0 & -a-2b & -a+b \\ 0 & 2b+c & -b-2c \\ 1 & -c+b & 3c \end{vmatrix}$ Expanding along C_1

$\Rightarrow \quad = (a+b+c)\{0-0+1[(-a-2b)(-b-2c)-(b-a)(2b+c)]\}$

$\Rightarrow \quad = (a+b+c)\{0-0+1(ab+2ac+2b^2+4bc+2ab+ac-2b^2-bc)\}$

$\Rightarrow \quad = (a+b+c)(3ab+3ac+3bc)$ $\therefore \Delta = 3(a+b+c)(ab+bc+ca) = RHS$

(l) LHS : Let $\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$ By $R_3 \to R_3 - 2R_2$

$\Rightarrow \quad = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 0 & 0 & -x \end{vmatrix}$ Taking x common from R_3

$\Rightarrow \quad = x\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 0 & 0 & -1 \end{vmatrix}$ By $R_2 \to R_2 - 4R_1$

$\Rightarrow \quad = x\begin{vmatrix} x+y & x & x \\ x & 0 & -2x \\ 0 & 0 & -1 \end{vmatrix}$ Taking x common from C_2

$\Rightarrow \quad = x^2\begin{vmatrix} x+y & 1 & x \\ x & 0 & -2x \\ 0 & 0 & -1 \end{vmatrix}$ Expanding along C_2

$\Rightarrow \quad = x^2\{-1(-x-0)+0-0\}$ $\therefore \Delta = x^3 = RHS$

(m) LHS : Let $\Delta = \begin{vmatrix} 0 & b^2a & c^2a \\ a^2b & 0 & c^2b \\ a^2c & b^2c & 0 \end{vmatrix}$

Taking a^2, b^2, c^2 common from C_1, C_2 & C_3 respectively

$\Rightarrow \quad = a^2b^2c^2\begin{vmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$\Rightarrow \quad = a^2b^2c^2\begin{vmatrix} 2a & a & a \\ 2b & 0 & b \\ 2c & c & 0 \end{vmatrix}$ Taking 2 common from C_1

$\Rightarrow \quad = 2a^2b^2c^2\begin{vmatrix} a & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix}$ Taking a, b, c common from R_1, R_2, R_3 respectively

$$\Rightarrow \quad = 2a^3b^3c^3\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \qquad \text{By } R_1 \to R_1 - R_2$$

$$\Rightarrow \quad = 2a^3b^3c^3\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \qquad \text{Expanding along } C_3$$

$$\Rightarrow \quad = 2a^3b^3c^3\left\{0 - 1(0-1) + 0\right\} = 2a^3b^3c^3 = \text{RHS}$$

(n) LHS : Let $\Delta = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \quad = \begin{vmatrix} 3(x+y) & 3(x+y) & 3(x+y) \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} \qquad \text{Taking } 3(x+y) \text{ common from } R_1$$

$$\Rightarrow \Delta = 3(x+y)\begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} \qquad \text{By } C_1 \to C_1 - C_2 \ \& \ C_2 \to C_2 - C_3$$

$$\Rightarrow \Delta = 3(x+y)\begin{vmatrix} 0 & 0 & 1 \\ 2y & -y & x+y \\ -y & 2y & x \end{vmatrix} \qquad \text{Taking } y \text{ common from } C_1 \ \& \ C_2 \text{ both}$$

$$\Rightarrow \Delta = 3y^2(x+y)\begin{vmatrix} 0 & 0 & 1 \\ 2 & -1 & x+y \\ -1 & 2 & x \end{vmatrix} \qquad \text{Now, expanding along } R_1$$

$$\Rightarrow \Delta = 3y^2(x+y)\left\{0 - 0 + 1(4-1)\right\} = 9y^2(x+y) = \text{RHS}$$

(o) LHS : Let $\Delta = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ By $R_1 \to R_1 - xR_2$

$$\Rightarrow \Delta = \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \qquad \text{Taking } (1-x^2) \text{ common from } R_1$$

$$\Rightarrow \Delta = (1-x^2)\begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \qquad \text{By } R_2 \to R_2 - xR_1$$

$$\therefore \Delta = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix} = \text{RHS}$$

(p) LHS : Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$\Rightarrow \quad = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & x+z & x+y \end{vmatrix}$ Taking 2 common from C_1

$\Rightarrow \quad = \begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$ By $C_1 \to C_1 - (C_2 + C_3)$

$\Rightarrow \quad = 2\begin{vmatrix} -a & c+a & a+b \\ -p & r+p & p+q \\ -x & z+x & x+y \end{vmatrix}$ By $C_2 \to C_2 + C_1$ & $C_3 \to C_3 + C_1$

$\Rightarrow \quad = 2\begin{vmatrix} -a & c & b \\ -p & r & q \\ -x & z & y \end{vmatrix}$ Taking (-1) common from C_1 and applying $C_2 \leftrightarrow C_3$

$\Rightarrow \quad = 2(-1)(-1)\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ $\therefore \Delta = \text{RHS}$

(q) LHS : Let $\Delta = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$\Rightarrow \quad = \begin{vmatrix} 2(a+b+c) & 0 & a+b+c \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ Taking $(a+b+c)$ common from R_1

$\Rightarrow \quad = (a+b+c)\begin{vmatrix} 2 & 0 & 1 \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ Expanding along R_1

$\Rightarrow \quad = (a+b+c)\{2(bc-c^2-bc+ab)-0+1(c^2-a^2-ab+ac-b^2+bc)\}$

$\Rightarrow \quad = (a+b+c)\left[2bc-2c^2-2bc+2ab+c^2-a^2-ab+ac-b^2+bc\right]$

$\Rightarrow \quad = (a+b+c)\left[-a^2-b^2-c^2+ab+ac+bc\right]$

$\Rightarrow \quad = -(a+b+c)\left[a^2+b^2+c^2-ab-bc-ca\right]$

$\Rightarrow \quad = -\left[a^3+b^3+c^3-3abc\right] = 3abc-a^3-b^3-c^3 = \text{RHS}$

(r) LHS : Let $\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Take a, b & c common from C_1, C_2 & C_3 respectively

$\Rightarrow \quad = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$\Rightarrow \quad = abc\begin{vmatrix} 2(a+b) & 2(b+c) & 2(a+c) \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$ Take 2 common from R_1

$\Rightarrow \quad = 2abc\begin{vmatrix} a+b & b+c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$ By $R_1 \to R_1 - R_2$

$\Rightarrow \quad = 2abc\begin{vmatrix} 0 & c & c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$ Take c common from R_1

$\Rightarrow \quad = 2abc^2\begin{vmatrix} 0 & 1 & 1 \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$ By $C_2 \to C_2 - C_3$

$\Rightarrow \quad = 2abc^2\begin{vmatrix} 0 & 0 & 1 \\ a+b & b-a & a \\ b & b & c \end{vmatrix}$ Now, expanding along R_1

$\Rightarrow \quad = 2abc^2\left\{0 - 0 + 1\left(ab + b^2 - b^2 + ab\right)\right\} = 2abc^2(2ab)$ $\quad \therefore \Delta = 4a^2b^2c^2 = RHS$

(s) LHS : Let $\Delta = \begin{vmatrix} x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$\Rightarrow \quad = \begin{vmatrix} x^2-1 & x-1 & 0 \\ 2x-2 & x-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ Taking $(x-1)$ common from R_1 & R_2 both

$\Rightarrow \quad = (x-1)^2\begin{vmatrix} x+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ By $R_1 \to R_1 - R_2$

$\Rightarrow \quad = (x-1)^2\begin{vmatrix} x-1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ Taking $(x-1)$ common from R_1

$\Rightarrow \quad = (x-1)^3\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ Now, expanding along R_1

$\Rightarrow \Delta = (x-1)^3\left\{1(1-0) - 0 + 0\right\} = (x-1)^3$ $\quad \therefore \Delta = (x-1)^3 = RHS$

(t) LHS : Let $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ By $C_1 \to aC_1, C_2 \to bC_2 \ \& \ C_3 \to cC_3$

$\Rightarrow \Delta = \dfrac{1}{abc}\begin{vmatrix} a(b+c) & ab & ac \\ ab & b(c+a) & bc \\ ac & bc & c(a+b) \end{vmatrix}$

Taking a, b, c common from R_1, R_2, R_3 respectively

$\Rightarrow \quad = \dfrac{abc}{abc}\begin{vmatrix} b+c & b & c \\ a & c+a & c \\ a & b & a+b \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$\Rightarrow \quad = \begin{vmatrix} 2(b+c) & b & c \\ 2(a+c) & c+a & c \\ 2(a+b) & b & a+b \end{vmatrix}$ Taking 2 common from C_1

$\Rightarrow \quad = 2\begin{vmatrix} b+c & b & c \\ a+c & c+a & c \\ a+b & b & a+b \end{vmatrix}$ By $C_1 \to C_1 - C_3$

$\Rightarrow \quad = 2\begin{vmatrix} b & b & c \\ a & c+a & c \\ 0 & b & a+b \end{vmatrix}$ By $C_2 \to C_2 - C_1$

$\Rightarrow \quad = 2\begin{vmatrix} b & 0 & c \\ a & c & c \\ 0 & b & a+b \end{vmatrix}$ Expanding along R_1

$\Rightarrow \quad = 2\{b(ac+bc-bc)-0+c(ab-0)\} = 2[abc+abc]$ $\therefore \Delta = 4abc = RHS$

(u) LHS : Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$ Taking a, b, c common from R_1, R_2, R_3 respectively

$\Rightarrow \Delta = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$ Taking a, b, c common from C_1, C_2, C_3 respectively

$\Rightarrow \Delta = abc \times abc\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ By $R_1 \to R_1 + R_2, R_2 \to R_2 + R_3$

$\Rightarrow \Delta = a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix}$ Expanding along R_1,

$\Rightarrow \Delta = a^2b^2c^2[0-0+2(2-0)] = 4a^2b^2c^2 = (2abc)^2 = RHS$

Note that the value of $\Delta = 4a^2b^2c^2$ is a perfect square.

(v) LHS : Let $\Delta=\begin{vmatrix}1 & x & x^2\\ x^2 & 1 & x\\ x & x^2 & 1\end{vmatrix}$ By $C_1\to C_1-xC_3$

$\Rightarrow \quad =\begin{vmatrix}1-x^3 & x & x^2\\ 0 & 1 & x\\ 0 & x^2 & 1\end{vmatrix}$ Taking $\left(1-x^3\right)$ common from C_1

$\Rightarrow \quad =\left(1-x^3\right)\begin{vmatrix}1 & x & x^2\\ 0 & 1 & x\\ 0 & x^2 & 1\end{vmatrix}$ By $R_2\to R_2-xR_3$

$\Rightarrow \quad =\left(1-x^3\right)\begin{vmatrix}1 & x & x^2\\ 0 & 1-x^3 & 0\\ 0 & x^2 & 1\end{vmatrix}$ Taking $\left(1-x^3\right)$ common from R_2

$\Rightarrow \quad =\left(1-x^3\right)^2\begin{vmatrix}1 & x & x^2\\ 0 & 1 & 0\\ 0 & x^2 & 1\end{vmatrix}$ Expanding along C_1

$\Rightarrow \quad =\left(1-x^3\right)^2\left\{1(1-0)-0+0\right\}$ $\therefore \Delta=\left(1-x^3\right)^2=\text{RHS}$

(w) LHS : Let $\Delta=\begin{vmatrix}(y+z)^2 & xy & zx\\ xy & (x+z)^2 & yz\\ xz & yz & (x+y)^2\end{vmatrix}$ By $R_1\to xR_1,\ R_2\to yR_2\ \&\ R_3\to zR_3$

$\therefore \Delta=\dfrac{1}{xyz}\begin{vmatrix}x(y+z)^2 & x^2y & x^2z\\ xy^2 & y(x+z)^2 & y^2z\\ xz^2 & yz^2 & z(x+y)^2\end{vmatrix}$

Taking x, y, z common from C_1, C_2 and C_3 respectively

$\Rightarrow \quad =\dfrac{xyz}{xyz}\begin{vmatrix}(y+z)^2 & x^2 & x^2\\ y^2 & (x+z)^2 & y^2\\ z^2 & z^2 & (x+y)^2\end{vmatrix}$ By $C_2\to C_2-C_1$, $C_3\to C_3-C_1$

$\Rightarrow \Delta=\begin{vmatrix}(y+z)^2 & x^2-(y+z)^2 & x^2-(y+z)^2\\ y^2 & (x+z)^2-y^2 & 0\\ z^2 & 0 & (x+y)^2-z^2\end{vmatrix}$ Take $x+y+z$ common from C_2 & C_3 both

$\Rightarrow \Delta=(x+y+z)^2\begin{vmatrix}(y+z)^2 & x-y-z & x-y-z\\ y^2 & x+z-y & 0\\ z^2 & 0 & x+y-z\end{vmatrix}$ By $R_1\to R_1-(R_2+R_3)$

$$\Rightarrow \Delta=(x+y+z)^2\begin{vmatrix}2yz & -2z & -2y\\ y^2 & x-y+z & 0\\ z^2 & 0 & x+y-z\end{vmatrix}$$ By $C_2 \to C_2+\frac{1}{y}C_1$ and $C_3 \to C_3+\frac{1}{z}C_1$

$$\Rightarrow \Delta=(x+y+z)^2\begin{vmatrix}2yz & 0 & 0\\ y^2 & x+z & \frac{y^2}{z}\\ z^2 & \frac{z^2}{y} & x+y\end{vmatrix}$$ Expanding along R_1

$$\Rightarrow \quad =(x+y+z)^2\left\{2yz\left(x^2+xy+zx+zy-zy\right)-0+0\right\}$$

$$\Rightarrow \Delta=(x+y+z)^2\,2yzx(x+y+z)=2xyz(x+y+z)^3=\text{RHS}$$

(x) Let $\Delta=\begin{vmatrix}a & a^2 & bc\\ b & b^2 & ca\\ c & c^2 & ab\end{vmatrix}$ By $R_1 \to aR_1,\ R_2 \to bR_2, R_3 \to cR_3$

$$\Rightarrow \quad =\frac{1}{abc}\begin{vmatrix}a^2 & a^3 & abc\\ b^2 & b^3 & cab\\ c^2 & c^3 & abc\end{vmatrix}$$ Take (abc) common from C_3,

$$\Rightarrow \quad =\frac{abc}{abc}\begin{vmatrix}a^2 & a^3 & 1\\ b^2 & b^3 & 1\\ c^2 & c^3 & 1\end{vmatrix}$$ By $C_1 \leftrightarrow C_2$

$$\Rightarrow \quad =-\begin{vmatrix}a^3 & a^2 & 1\\ b^3 & b^2 & 1\\ c^3 & c^2 & 1\end{vmatrix}$$ By $C_1 \leftrightarrow C_3$

$$\Rightarrow \quad =\begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}\ldots(i)$$

Now applying $R_1 \to R_1-R_2, R_2 \to R_2-R_3$, we have

$$\Delta=\begin{vmatrix}0 & a^2-b^2 & a^3-b^3\\ 0 & b^2-c^2 & b^3-c^3\\ 1 & c^2 & c^3\end{vmatrix}$$ Take $(a-b)$ & $(b-c)$ common from R_1 & R_2 respectively

$$\Rightarrow \quad =(a-b)(b-c)\begin{vmatrix}0 & a+b & a^2+ab+b^2\\ 0 & b+c & b^2+bc+c^2\\ 1 & c^2 & c^3\end{vmatrix}$$ By $R_1 \to R_1-R_2$

$$\Rightarrow \quad =(a-b)(b-c)\begin{vmatrix}0 & a-c & a^2-c^2+ab-bc\\ 0 & b+c & b^2+bc+c^2\\ 1 & c^2 & c^3\end{vmatrix}$$ Take $(c-a)$ common from R_1,

$$\Rightarrow \quad =(a-b)(b-c)(c-a)\begin{vmatrix}0 & -1 & -a-c-b\\0 & b+c & b^2+bc+c^2\\1 & c^2 & c^3\end{vmatrix} \qquad \text{Expanding along } C_1,$$

$$\Rightarrow \Delta=(a-b)(b-c)(c-a)\left[0-0+1\left(-b^2-bc-c^2+ab+bc+b^2+ac+c^2+bc\right)\right]$$

$$\Rightarrow \Delta=(a-b)(b-c)(c-a)(ab+bc+ca)\ldots\text{(ii)}$$

Hence by (i) & (ii), $\begin{vmatrix}a & a^2 & bc\\b & b^2 & ca\\c & c^2 & ab\end{vmatrix}=\begin{vmatrix}1 & a^2 & a^3\\1 & b^2 & b^3\\1 & c^2 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca)$

(y) LHS : Let $\Delta=\begin{vmatrix}a+b+c & -c & -b\\-c & a+b+c & -a\\-b & -a & a+b+c\end{vmatrix}$ By $R_1\to R_1+R_2+R_3$

$$\Rightarrow \quad =\begin{vmatrix}a & b & c\\-c & a+b+c & -a\\-b & -a & a+b+c\end{vmatrix} \qquad \text{By } R_1\to R_1-R_2,\ R_2\to R_2+R_3$$

$$\Rightarrow \quad =\begin{vmatrix}a+c & -(a+c) & a+c\\-(b+c) & b+c & b+c\\-b & -a & a+b+c\end{vmatrix}$$

Taking $(a+c)$ & $(b+c)$ common from R_1 & R_2 respectively.

$$\Rightarrow \quad =(a+c)(b+c)\begin{vmatrix}1 & -1 & 1\\-1 & 1 & 1\\-b & -a & a+b+c\end{vmatrix} \qquad \text{By } R_1\to R_1+R_2$$

$$\Rightarrow \quad =(a+c)(b+c)\begin{vmatrix}0 & 0 & 2\\-1 & 1 & 1\\-b & -a & a+b+c\end{vmatrix} \qquad \text{Expanding along } R_1$$

$$\therefore \Delta=(a+c)(b+c)\{0-0+2(a+b)\}=2(a+b)(b+c)(c+a)=\text{RHS}$$

(z) LHS : Let $\Delta=\begin{vmatrix}x+y+2z & x & y\\z & y+z+2x & y\\z & x & z+x+2y\end{vmatrix}$ By $C_1\to C_1+C_2+C_3$

$$\Rightarrow \quad =\begin{vmatrix}2(x+y+z) & x & y\\2(x+y+z) & y+z+2x & y\\2(x+y+z) & x & z+x+2y\end{vmatrix} \qquad \text{Taking } 2(x+y+z) \text{ common from } C_1$$

$$\Rightarrow \quad =2(x+y+z)\begin{vmatrix}1 & x & y\\1 & y+z+2x & y\\1 & x & z+x+2y\end{vmatrix} \qquad \text{By } R_1\to R_1-R_2\ \& R_2\to R_2-R_3$$

$$\Rightarrow \quad =2(x+y+z)\begin{vmatrix}0 & -(x+y+z) & 0\\0 & x+y+z & -(x+y+z)\\1 & x & z+x+2y\end{vmatrix}$$

Taking $(x+y+z)$ common from R_1 & R_2 both

$$\Rightarrow \quad = 2(x+y+z)^3 \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & x & z+x+2y \end{vmatrix} \qquad \text{Expanding along } R_1$$

$$\Rightarrow \Delta = 2(x+y+z)^3\{0+1(0+1)+0\} = 2(x+y+z)^3 = \text{RHS}$$

(aa) LHS : Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix}$ By $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$\Rightarrow \Delta = \begin{vmatrix} x-y & y-z & z \\ (x-y)(x+y) & (y-z)(y+z) & z^2 \\ -(x-y) & -(y-z) & x+y \end{vmatrix}$$

Taking $(x-y)$ & $(y-z)$ common from C_1 & C_2 respectively.

$$\Rightarrow \quad = (x-y)(y-z)\begin{vmatrix} 1 & 1 & z \\ (x+y) & (y+z) & z^2 \\ -1 & -1 & x+y \end{vmatrix} \qquad \text{By } C_1 \to C_1 - C_2$$

$$\Rightarrow \quad = (x-y)(y-z)\begin{vmatrix} 0 & 1 & z \\ -(z-x) & (y+z) & z^2 \\ 0 & -1 & (x+y) \end{vmatrix} \qquad \text{Take } (z-x) \text{ common from } C_1$$

$$\Rightarrow \quad = (x-y)(y-z)(z-x)\begin{vmatrix} 0 & 1 & z \\ -1 & (y+z) & z^2 \\ 0 & -1 & (x+y) \end{vmatrix} \qquad \text{Expanding along } C_1$$

$$\Rightarrow \quad = (x-y)(y-z)(z-x)\{0+1(x+y+z)+0\}$$

$$\Rightarrow \quad = (x-y)(y-z)(z-x)(x+y+z) = \text{LHS}$$

(ab) LHS : Let $\Delta = \begin{vmatrix} 1+x^2-y^2 & 2xy & -2y \\ 2xy & 1-x^2+y^2 & 2x \\ 2y & -2x & 1-x^2-y^2 \end{vmatrix}$ By $C_1 \to C_1 - yC_3$

$$\Rightarrow \quad = \begin{vmatrix} 1+x^2+y^2 & 2xy & -2y \\ 0 & 1-x^2+y^2 & 2x \\ 2y-y(1-x^2-y^2) & -2x & 1-x^2-y^2 \end{vmatrix} \qquad \text{By } C_2 \to C_2 + xC_3$$

$$\Rightarrow \quad = \begin{vmatrix} 1+x^2+y^2 & 0 & -2y \\ 0 & 1+x^2+y^2 & 2x \\ 2y-y+yx^2+y^3 & -2x+x-x^3-xy^2 & 1-x^2-y^2 \end{vmatrix}$$

$$\Rightarrow \quad = \begin{vmatrix} 1+x^2+y^2 & 0 & -2y \\ 0 & 1+x^2+y^2 & 2x \\ y(1+x^2+y^2) & -x(1+x^2+y^2) & 1-x^2-y^2 \end{vmatrix}$$

Taking $(a+x^2+y^2)$ common from C_1 & C_2 both

$$\Rightarrow \quad = \left(1+x^2+y^2\right)^2 \begin{vmatrix} 1 & 0 & -2y \\ 0 & 1 & 2x \\ y & -x & 1-x^2-y^2 \end{vmatrix} \qquad \text{By } R_3 \to R_3 + xR_2 - yR_2$$

$$\Rightarrow \quad = \left(1+x^2+y^2\right)^2 \begin{vmatrix} 1 & 0 & -2y \\ 0 & 1 & 2x \\ 0 & 0 & 1+x^2+y^2 \end{vmatrix} \qquad \text{Taking } \left(1+x^2+y^2\right) \text{ common from } R_3$$

$$\Rightarrow \quad = \left(1+x^2+y^2\right)^3 \begin{vmatrix} 1 & 0 & -2y \\ 0 & 1 & 2x \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{Now expanding along } R_1$$

$$\Rightarrow \quad = \left(1+x^2+y^2\right)^3 \left\{1(1-0)-0-2y(0-0)\right\} = \left(1+x^2+y^2\right)^3 = \text{RHS}$$

(ac) LHS : Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Take a, b & c common from R_1, R_2 & R_3 respectively

$$\Rightarrow \quad = abc \begin{vmatrix} 1+1/a & 1/a & 1/a \\ 1/b & 1+1/b & 1/b \\ 1/c & 1/c & 1+1/c \end{vmatrix} \qquad \text{By } R_1 \to R_1 + R_2 + R_3$$

$$\Rightarrow \quad = abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1, $\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$

By $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$, $\Delta = \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) abc \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & 1+\frac{1}{b} \end{vmatrix}$

Now expanding along C_1, $\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left[0+1(0+1)+0\right] = abc\left(a+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ldots$(i)

Also, $\Delta = abc + bc + ca + ab \ldots$(ii)

Hence by (i) & (ii), $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab$

(ad) LHS : Let $\Delta = \begin{vmatrix} a+b+nc & na-a & nb-b \\ nc-c & b+c+na & nb-b \\ nc-c & na-a & c+a+nb \end{vmatrix}$ By $C_1 \to C_2 + C_2 + C_3$

$\Rightarrow \quad = \begin{vmatrix} n(a+b+c) & na-a & nb-b \\ n(a+b+c) & b+c+na & nb-b \\ n(a+b+c) & na-a & c+a+nb \end{vmatrix}$ Taking $n(a+b+c)$ common from C_1

$\Rightarrow \quad = n(a+b+c)\begin{vmatrix} 1 & na-a & nb-b \\ 1 & b+c+na & nb-b \\ 1 & na-a & c+a+nb \end{vmatrix}$ By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$\Rightarrow \quad = n(a+b+c)\begin{vmatrix} 0 & -(a+b+c) & 0 \\ 0 & a+b+c & -(a+b+c) \\ 1 & na-a & c+a+nb \end{vmatrix}$

Taking $a+b+c$ common from R_1 and R_2 both

$\Rightarrow \quad = n(a+b+c)^3\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & na-a & (c+a+nb) \end{vmatrix}$ Expanding along C_1

$\Rightarrow \quad = n(a+b+c)^3\{0-0+1(1-0)\} = n(a+b+c)^3 = \text{RHS}$

(ae) LHS: Let $\Delta = \begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix}$ By $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$\Rightarrow \Delta = \begin{vmatrix} (x-2)^2-(x-1)^2 & (x-1)^2-x^2 & x^2 \\ (x-1)^2-x^2 & x^2-(x+1)^2 & (x+1)^2 \\ x^2-(x+1)^2 & (x+1)^2-(x+2)^2 & (x+2)^2 \end{vmatrix}$

$\Rightarrow \Delta = \begin{vmatrix} (2x-3)(-1) & (2x-1)(-1) & x^2 \\ (2x-1)(-1) & (2x+1)(-1) & (x+1)^2 \\ (2x+1)(-1) & (2x+3)(-1) & (x+2)^2 \end{vmatrix}$ Taking '–1' common from both C_1 & C_2

$\Rightarrow \Delta = (-1)(-1)\begin{vmatrix} (2x-3) & (2x-1) & x^2 \\ (2x-1) & (2x+1) & (x+1)^2 \\ (2x+1) & (2x+3) & (x+2)^2 \end{vmatrix}$ By $C_1 \to C_1 - C_2$

$\Rightarrow \Delta = \begin{vmatrix} -2 & (2x-1) & x^2 \\ -2 & (2x+1) & (x+1)^2 \\ -2 & (2x+3) & (x+2)^2 \end{vmatrix}$ Taking '–2' common from C_1

$$\Rightarrow \Delta = -2\begin{vmatrix} 1 & (2x-1) & x^2 \\ 1 & (2x+1) & (x+1)^2 \\ 1 & (2x+3) & (x+2)^2 \end{vmatrix}$$ By $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_2$

$$\Rightarrow \Delta = -2\begin{vmatrix} 1 & (2x-1) & x^2 \\ 0 & 2 & 2x+1 \\ 0 & 2 & 2x+3 \end{vmatrix}$$ Expanding along C_1,

$$\Rightarrow \Delta = -2\left[1(4x+6-4x-2)-0+0\right] = -8 = RHS$$

(af) LHS : Let $\Delta = \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad = \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix}$$ Taking $(x+a+b+c)$ common from R_1

$$\Rightarrow \quad = (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$ By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$\Rightarrow \quad = (x+a+b+c)\begin{vmatrix} 0 & -x & 0 \\ 0 & x & -x \\ 1 & b & x+c \end{vmatrix}$$ Taking 'x' common from R_1 & R_2 both

$$\Rightarrow \quad = x^2(x+a+b+c)\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & b & (x+c) \end{vmatrix}$$ Expanding along C_1

$$\Rightarrow \quad = x^2(x+a+b+c)\{0-0+1(1-0)\} = x^2(x+a+b+c) = RHS$$

(ag) LHS : Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$ By $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$\Rightarrow \Delta = \begin{vmatrix} x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \\ z(y-x) & x(z-y) & xy \end{vmatrix}$$

Take $(x-y)$ and $(y-z)$ common from C_1 and C_2 respectively

$$\Rightarrow \Delta = (x-y)(y-z)\begin{vmatrix} 1 & 1 & z \\ x+y & y+z & z^2 \\ -z & -x & xy \end{vmatrix}$$ By $C_2 \to C_2 - C_1$

$$\Rightarrow \Delta = (x-y)(y-z)\begin{vmatrix} 1 & 0 & z \\ x+y & z-x & z^2 \\ -z & z-x & xy \end{vmatrix}$$ Taking $(z-x)$ common from C_2,

$$\Rightarrow \Delta = (x-y)(y-z)(z-x)\begin{vmatrix} 1 & 0 & z \\ x+y & 1 & z^2 \\ -z & 1 & xy \end{vmatrix}$$ Expanding along R_1,

$$\Rightarrow \Delta = (x-y)(y-z)(z-x)\left[1(xy-z^2)-0+z(x+y+z)\right]$$

$$\Rightarrow \Delta = (x-y)(y-z)(z-x)\left[xy-z^2+zx+yz+z^2\right]$$

$$\therefore \Delta = (x-y)(y-z)(z-x)(xy+yz+zx) = \text{RHS}$$

(ah) LHS : Let $\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$ By $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} ab^2+ac^2 & a^2b & a^2c \\ b^2a & bc^2+a^2b & b^2c \\ c^2a & c^2b & ca^2+cb^2 \end{vmatrix}$$ $\left[\text{Taking } a,b,c \text{ common from } C_1, C_2, C_3 \text{ respectively}\right.$

$$\Rightarrow \Delta = \frac{abc}{abc}\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$ By $R_1 \to R_1+R_2+R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 2(b^2+c^2) & 2(a^2+c^2) & 2(a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$ Taking 2 common from R_1,

$$\Rightarrow \Delta = 2\begin{vmatrix} b^2+c^2 & a^2+c^2 & a^2+b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$ By $R_2 \to R_2-R_1, R_3 \to R_3-R_1$

$$\Rightarrow \Delta = 2\begin{vmatrix} b^2+c^2 & a^2+c^2 & a^2+b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$ By $R_1 \to R_1+R_2+R_3$

$$\Rightarrow \Delta = 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$ Expanding along R_1,

$$\Rightarrow \Delta = 2\left[0-c^2\left(0-a^2b^2\right)+b^2\left(a^2c^2-0\right)\right]$$

$$\Rightarrow \Delta = 2\left[a^2b^2c^2+a^2b^2c^2\right] = 4a^2b^2c^2 = \text{RHS}$$

(ai) LHS : Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$ By $R_2 \to R_2-R_1, R_3 \to R_3-R_1$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix}$$ Taking (a – b) and (c – a) common from R_2 & R_3 respectively

$$\Rightarrow \Delta = (a-b)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & -1 & -b^2-a^2-ab \\ 0 & 1 & c^2+a^2+ac \end{vmatrix} \qquad \text{By } R_2 \to R_2 + R_3$$

$$\Rightarrow \Delta = (a-b)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & c^2-b^2-ab+ac \\ 0 & 1 & c^2+a^2+ac \end{vmatrix} = (a-b)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & -(b-c)(c+b+a) \\ 0 & 1 & c^2+a^2+ac \end{vmatrix}$$

Taking $(b-c)(c+b+a)$ common from R_2,

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & -1 \\ 0 & 1 & c^2+a^2+ac \end{vmatrix} \qquad \text{Expanding along } C_1,$$

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)(a+b+c)\left[1(0+1)-0+0\right]$$

$$\therefore \Delta = (a-b)(b-c)(c-a)(a+b+c) = \text{RHS}$$

(aj) LHS : Let $\Delta = \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 2(b^2+c^2) & 2(a^2+c^2) & 2(a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \qquad \text{Taking 2 common from } R_1,$$

$$\Rightarrow \Delta = 2\begin{vmatrix} b^2+c^2 & a^2+c^2 & a^2+b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \qquad \text{By } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\Rightarrow \Delta = 2\begin{vmatrix} b^2+c^2 & a^2+c^2 & a^2+b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix} \qquad \text{By } R_1 \to R_1 + R_2 + R_3$$

$$\Rightarrow \Delta = 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix} \qquad \text{Expanding along } R_1,$$

$$\Rightarrow \Delta = 2\left[0 - c^2\left(0 - a^2b^2\right) + b^2\left(a^2c^2 - 0\right)\right]$$

$$\Rightarrow \Delta = 2\left[a^2b^2c^2 + a^2b^2c^2\right] = 4a^2b^2c^2 = \text{RHS}$$

(ak) LHS : Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ Apply $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$

$$\Rightarrow \Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix} \qquad \text{Take } (a+b+c) \text{ common from } C_1 \text{ \& } C_2 \text{ both}$$

$$\Rightarrow \Delta = (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \quad \text{Apply } R_3 \to R_3 - (R_1+R_2)$$

$$\Rightarrow \Delta = (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad \text{Apply } C_1 \to C_1(a) \;\&\; C_2 \to C_2(b)$$

$$\Rightarrow \Delta = \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ca-a^2 & 0 & a^2 \\ 0 & bc+ab-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix} \quad \text{Apply } C_1 \to C_1 + C_3,\ C_2 \to C_2 + C_3$$

$$\Rightarrow \Delta = \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ca & a^2 & a^2 \\ b^2 & bc+ab & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \quad \left[\begin{array}{l}\text{Take a, b, 2ab common} \\ \text{from } R_1, R_2, R_3 \text{ respectively.}\end{array}\right.$$

$$\Rightarrow \Delta = \frac{2a^2b^2(a+b+c)^2}{ab}\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = 2ab(a+b+c)^2\left[0-0+1\left(bc+ab+c^2+ac-ab\right)\right]$$

$$\Rightarrow \Delta = 2abc(a+b+c)^2(a+b+c) \qquad \therefore \Delta = 2abc(a+b+c)^3 = RHS$$

(al) LHS : Let $\Delta = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$ [Applying $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$

$= \begin{vmatrix} x+y+z & -y-z-x & 2y \\ 0 & x+y+z & z-x-y \\ -x-y-z & 0 & 2x \end{vmatrix}$ [Taking x + y + z common from C_1 and C_2 both

$= (x+y+z)^2\begin{vmatrix} 1 & -1 & 2y \\ 0 & 1 & z-x-y \\ -1 & 0 & 2x \end{vmatrix}$ [Applying $R_1 \to R_1 + R_2$

$= (x+y+z)^2\begin{vmatrix} 1 & 0 & y+z-x \\ 0 & 1 & z-x-y \\ -1 & 0 & 2x \end{vmatrix}$ [Expanding along R_1

$$= (x+y+z)^2\left[1\begin{vmatrix} 1 & z-x-y \\ 0 & 2x \end{vmatrix} - 0\begin{vmatrix} 0 & z-x-y \\ -1 & 2x \end{vmatrix} + (y+z-x)\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}\right]$$

$$= (x+y+z)^2[(2x-0)-0+(y+z-x)(0+1)]$$

$$= (x+y+z)^3 = RHS.$$

(am) LHS : Let $\Delta = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix}$$ [Taking $(a+x+y+z)$ common from C_1

$$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$= (a+x+y+z)\begin{vmatrix} 0 & -a & 0 \\ 0 & a & -a \\ 1 & y & a+z \end{vmatrix}$$ [Expanding along R_1

$$= (a+x+y+z)\left[0-(-a)\begin{vmatrix} 0 & -a \\ 1 & a+z \end{vmatrix}+0\right]$$

$$= a^2(a+x+y+z) = \text{RHS}$$

(an) LHS : Let $\Delta = \begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 5x+\lambda & 2x & 2x \\ 5x+\lambda & x+\lambda & 2x \\ 5x+\lambda & 2x & x+\lambda \end{vmatrix}$$ [Taking $(5x+\lambda)$ common from C_1

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+\lambda & 2x \\ 1 & 2x & x+\lambda \end{vmatrix}$$ [Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x+\lambda & 0 \\ 0 & 0 & -x+\lambda \end{vmatrix}$$ [Taking $(\lambda - x)$ common from R_2 & R_3 both

$$= (5x+\lambda)(\lambda-x)^2\begin{vmatrix} 1 & 2x & 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ [Expanding along C_1

$$= (5x+\lambda)(\lambda-x)^2\left[1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}-0+0\right]$$

$$= (5x+\lambda)(\lambda-x)^2 = \text{RHS}.$$

(ao) LHS : $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 3 & 4+3p & 2+4p+3q \\ 4 & 7+4p & 2+7p+4q \end{vmatrix}$ By $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$

$\Rightarrow \quad = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 3 & -2+3p \end{vmatrix}$ By $R_3 \to R_3 - 3R_2$

$$\Rightarrow \quad = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{Expanding along } C_1,$$

$$\Rightarrow \quad = 1(1-0)-0+0 = 1 = \text{RHS}$$

(ap) LHS : Let $\Delta = \begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix}$ By $R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = \begin{vmatrix} a^3-b^3 & 0 & a-b \\ b^3-c^3 & 0 & b-c \\ c^3 & 2 & c \end{vmatrix} \qquad \text{Taking } (a-b)\ \&\ (b-c) \text{ common from } R_1 \text{ and } R_2 \text{ respectively}$$

$$\Rightarrow \Delta = (a-b)(b-c)\begin{vmatrix} a^2+ab+b^2 & 0 & 1 \\ b^2+bc+c^2 & 0 & 1 \\ c^3 & 2 & c \end{vmatrix} \qquad \text{By } R_1 \to R_1 - R_2$$

$$\Rightarrow \Delta = (a-b)(b-c)\begin{vmatrix} (a-c)(a+c)+b(a-c) & 0 & 0 \\ b^2+bc+c^2 & 0 & 1 \\ c^3 & 2 & c \end{vmatrix} \qquad \text{Taking } (c-a) \text{ common from } R_1$$

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)\begin{vmatrix} -a-b-c & 0 & 0 \\ b^2+bc+c^2 & 0 & 1 \\ c^3 & 2 & c \end{vmatrix} \qquad \text{Expanding along } R_1$$

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)\{(-a-b-c)[0-2]-0+0\}$$

$$\therefore \Delta = 2(a-b)(b-c)(c-a)(a+b+c) = \text{RHS}$$

(aq) LHS : Let $\Delta = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x-1)(x+1) \end{vmatrix}$

Taking x from R_2, $x(x-1)$ from R_3 & $(x+1)$ from C_3

$$\Rightarrow \Delta = x^2(x^2-1)\begin{vmatrix} 1 & x & 1 \\ 2 & (x-1) & 1 \\ -3 & (x-2) & 1 \end{vmatrix} \qquad \text{By } C_2 \to C_2 - xC_1,\ C_3 \to C_3 - C_1.$$

$$\Rightarrow \Delta = x^2(x^2-1)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -x-1 & -1 \\ -3 & 4x-2 & 4 \end{vmatrix} \qquad \text{Expanding along } R_1$$

$$\Rightarrow \Delta = x^2(x^2-1)\left[1\left((-x-1)4+(4x-2)\right)-0+0\right] \qquad \Rightarrow \Delta = 6x^2(1-x^2) = \text{RHS}.$$

Q04. LHS: Let $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

$$\Rightarrow \quad = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

In det.(II), taking x, y, z common from R_1, R_2, R_3 respectively and 'p' from C_3

$$\Rightarrow \quad = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad \text{By } C_3 \leftrightarrow C_2 \ \& C_2 \leftrightarrow C_1 \text{ in det (I)}$$

$$\Rightarrow \quad = (-1)(-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Rightarrow \quad = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+pxyz) \qquad \text{By } R_1 \to R_1 - R_2 \ \& \ R_2 \to R_2 - R_3$$

$$\Rightarrow \quad = (1+pxyz)\begin{vmatrix} 0 & x-y & (x-y)(x+y) \\ 0 & y-z & (y-z)(y+z) \\ 1 & z & z^2 \end{vmatrix}$$

Taking $(x-y)$ & $(y-z)$ common from R_1 & R_2 respectively.

$$\Rightarrow \quad = (1+pxyz)(x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix} \qquad \text{By } R_1 \to R_1 - R_2$$

$$\Rightarrow \quad = (1+pxyz)(x-y)(y-z)\begin{vmatrix} 0 & 0 & -(z-x) \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix} \qquad \text{Taking } (z-x) \text{ common from } R_1$$

$$\Rightarrow \quad = (1+pxyz)(x-y)(y-z)(z-x)\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix} \qquad \text{Now, expanding along } C_1$$

$$\Rightarrow \quad = (1+pxyz)(x-y)(y-z)(z-x)\{0-0+1(0+1)\}$$

$$\therefore \Delta = (1+pxyz)(x-y)(y-z)(z-x) = \text{RHS}$$

Q05. We have $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} = 0$$

In det (II), taking x, y, z common from R_1, R_2, R_3 respectively and 'p' from C_3

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \qquad \text{By } C_3 \leftrightarrow C_2 \ \& C_2 \leftrightarrow C_1 \text{ in det (I)}$$

$$\Rightarrow (-1)(-1)\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}+pxyz\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}=0 \Rightarrow \begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}(1+pxyz)=0$$

By $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$, we've : $(1+pxyz)\begin{vmatrix}0 & x-y & (x-y)(x+y)\\0 & y-z & (y-z)(y+z)\\1 & z & z^2\end{vmatrix}=0$

Taking $(x-y)$ & $(y-z)$ common from R_1 & R_2 respectively.

$$\Rightarrow (1+pxyz)(x-y)(y-z)\begin{vmatrix}0 & 1 & x+y\\0 & 1 & y+z\\1 & z & z^2\end{vmatrix}=0 \qquad \text{By } R_1 \to R_1 - R_2$$

$$\Rightarrow (1+pxyz)(x-y)(y-z)\begin{vmatrix}0 & 0 & -(z-x)\\0 & 1 & y+z\\1 & z & z^2\end{vmatrix}=0 \qquad \text{Taking } (z-x) \text{ common from } R_1$$

$$\Rightarrow (1+pxyz)(x-y)(y-z)(z-x)\begin{vmatrix}0 & 0 & -1\\0 & 1 & y+z\\1 & z & z^2\end{vmatrix}=0 \qquad \text{Now, expanding along } C_1$$

$$\Rightarrow (1+pxyz)(x-y)(y-z)(z-x)\{0-0+1(0+1)\}=0$$

$$\Rightarrow (1+pxyz)(x-y)(y-z)(z-x)=0$$

Since it is known that $x \neq y \neq z$, therefore $(1+pxyz)=0$.

Q06. LHS : Let $\Delta=\begin{vmatrix}a & b-c & b+c\\a+c & b & c-a\\a-b & b+a & c\end{vmatrix}$ Applying $C_1 \to aC_1$

$$\Rightarrow \Delta=\frac{1}{a}\begin{vmatrix}a^2 & b-c & b+c\\a^2+ac & b & c-a\\a^2-ab & b+a & c\end{vmatrix} \qquad \text{Applying } C_1 \to C_1+bC_2+cC_3$$

$$\Rightarrow \Delta=\frac{1}{a}\begin{vmatrix}a^2+b^2+c^2 & b-c & b+c\\a^2+b^2+c^2 & b & c-a\\a^2+b^2+c^2 & b+a & c\end{vmatrix} \qquad \text{Taking } (a^2+b^2+c^2) \text{ common from } C_1,$$

$$\Rightarrow \Delta=\frac{a^2+b^2+c^2}{a}\begin{vmatrix}1 & b-c & b+c\\1 & b & c-a\\1 & b+a & c\end{vmatrix} \qquad \text{By } R_1 \to R_1-R_2,\ R_2 \to R_2-R_3$$

$$\Rightarrow \Delta=\frac{a^2+b^2+c^2}{a}\begin{vmatrix}0 & -c & b+a\\0 & -a & -a\\1 & b+a & c\end{vmatrix} \qquad \text{Expanding along } C_1,$$

$$\Rightarrow \Delta=\frac{a^2+b^2+c^2}{a}\left[0-0+1\left(ac+ab+a^2\right)\right]$$

$\therefore \Delta=(a+b+a)(a^2+b^2+c^2)=RHS$

Q07. Given that a, b, c are p^{th}, q^{th} and r^{th} terms of a G.P. then,

$A_p = AR^{p-1} = a, A_q = AR^{q-1} = b, A_r = AR^{r-1} = c$, where A and R are the 1st term & common ratio of the geometric progression.

Consider LHS : Let $\Delta = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$ $\Rightarrow \Delta = \begin{vmatrix} \log\left[AR^{p-1}\right] & p & 1 \\ \log\left[AR^{q-1}\right] & q & 1 \\ \log\left[AR^{r-1}\right] & r & 1 \end{vmatrix}$

$\Rightarrow \Delta = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$ By $C_1 \to C_1 - [\log A]C_3$

$\Rightarrow \Delta = \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix}$ Taking log R common from C_1,

$\Rightarrow \Delta = \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$ By $C_1 \to C_1 + C_3$

$\Rightarrow \Delta = \log R \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix}$ Since C_1 & C_2 are identical, $\therefore \Delta = 0 = RHS$

Q08. (a) Given that a, b, c are in A.P. so, 2b = a + c …(i)

Consider LHS : Let $\Delta = \begin{vmatrix} x+1 & x+3 & x+a \\ x+2 & x+5 & x+b \\ x+3 & x+7 & x+c \end{vmatrix}$ By $R_1 \to R_1 + R_3 - 2R_2$

$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & a+c-2b \\ x+2 & x+5 & x+b \\ x+3 & x+7 & x+c \end{vmatrix}$ $[\because a+c = 2b \quad \Rightarrow a+c-2b = 0$

$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+2 & x+5 & x+b \\ x+3 & x+7 & x+c \end{vmatrix}$ Since all the elements of R_1 are 0, $\therefore \Delta = 0 = RHS$

(b) Proceed same as in last sum. Value of determinant = 0

Q09. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$\Rightarrow \quad = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$ Taking $(a+b+c)$ common R_1

$\Rightarrow \quad = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$ By $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$\Rightarrow \quad = (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$ Now, expanding along R_1

$\Rightarrow \quad = (a+b+c)\{0-0+1(ab-b^2-ac+bc-c^2+ac+ac-a^2)\}$

$\Rightarrow \quad = -(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

$\therefore \Delta = -\frac{1}{2}(a+b+c)[2a^2+2b^2+2c^2-2ab-2bc-2ca]$

$\Rightarrow \quad = -\frac{1}{2}(a+b+c)\{a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ca\}$

$\Rightarrow \quad = -\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$

$\because$ a, b, c are all positive, so $a+b+c>0$

And also a, b, c are unequal so, $(a-b)^2+(b-c)^2+(c-a)^2>0$.

That is, $\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}>0$

$\Rightarrow -\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}<0 \qquad \therefore \Delta<0$

Hence $\Delta<0$ i.e., the value of given determinant is negative.

Q10. **(a)** We have $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=0$ By $R_1 \to R_1+R_2+R_3$

$\Rightarrow \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=0$ Take $2(a+b+c)$ common from C_1,

$\Rightarrow 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=0$ By $C_1 \to C_1-C_2, C_2 \to C_2-C_3$

$\Rightarrow 2(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ c-b & a-c & b+c \\ a-c & b-a & c+a \end{vmatrix}=0$ Expanding along R_1,

$\Rightarrow 2(a+b+c)\left[0-0+1\{(c-b)(b-a)-(a-c)^2\}\right]=0$

$\Rightarrow 2(a+b+c)\{bc-ac-b^2+ab+2ac-c^2-a^2\}=0$

$\Rightarrow -2(a+b+c)\{a^2+b^2+c^2-ab-bc-ca\}=0$

$\Rightarrow -(a+b+c)\{2a^2+2b^2+2c^2-2ab-2bc-2ca\}=0$

$\Rightarrow -(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}=0$

$\therefore (a+b+c)=0$ or, $(a-b)^2+(b-c)^2+(c-a)^2=0$

Since a, b and c are real numbers and, $(a-b)^2+(b-c)^2+(c-a)^2=0$ implies that,

$(a-b)^2=0, (b-c)^2=0, (c-a)^2=0 \qquad \Rightarrow a=b, b=c, c=a$

Hence, $a+b+c=0$ or, $a=b=c$.

(b) We have $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$ By $C_1 \to C_1+C_2+C_3$

$\Rightarrow \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}=0$ Taking $a+b+c$ common from C_1

$\Rightarrow (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}=0$ By $R_1 \to R_1-R_2, R_2 \to R_2-R_3$

$\Rightarrow (a+b+c)\begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}=0$ Expanding along C_1,

$\Rightarrow (a+b+c)\left[0-0+1.\left((b-c)(a-b)-(c-a)^2\right)\right]=0$

$\Rightarrow (a+b+c)\left(ab-b^2-ac+bc-c^2+2ac-a^2\right)=0$

$\because a+b+c \neq 0$ so, $\left(ab+bc+ca-a^2-b^2-c^2\right)=0$

Now $\because a+b+c \neq 0$ so, $\left(ab+bc+ca-a^2-b^2-c^2\right)=0$ $\Rightarrow -\left(a^2+b^2+c^2-ab-ac-bc\right)=0$

$\Rightarrow \frac{1}{2}\left[2a^2+2b^2+2c^2-2ab-2bc-2ca\right]=0$

$\Rightarrow \frac{1}{2}\left\{a^2+b^2-2ab+b^2+c^2-2bc+c^2+a^2-2ca\right\}=0$

$\Rightarrow \frac{1}{2}\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}=0$

$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$ $\Rightarrow (a-b)^2=0, (b-c)^2=0, (c-a)^2=0$

$\Rightarrow a=b, b=c, c=a$. Therefore, $a=b=c$. Hence proved.

(c) Proceed same as in Q10 (b) and obtain $\Delta=(a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right)=0$

$\Rightarrow \Delta=-\frac{1}{2}(a+b+c)\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}=0$ $\because a \neq b \neq c$ $\therefore a+b+c=0$.

Q11. We have $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0, a \neq 0$ By $R_1 \to R_1+R_2+R_3$

$\Rightarrow \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$ Taking $(3x+a)$ common from R_1,

$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$ By $C_1 \to C_1-C_2$, $C_2 \to C_2-C_3$

$$\Rightarrow (3x+a)\begin{vmatrix} 0 & 0 & 1 \\ -a & a & x \\ 0 & -a & x+a \end{vmatrix} = 0$$ Expanding along R_1,

$$\Rightarrow (3x+a)\left[0-0+1\left(a^2-0\right)\right]=0$$ $$\Rightarrow (3x+a)a^2 = 0$$

Since $a \neq 0, \therefore x = -\frac{a}{3}$.

Q12. We have $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} 3a-x & 3a-x & 3a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$ Taking $(3a-x)$ common from R_1,

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & 1 & 1 \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$ By $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$\Rightarrow (3a-x)\begin{vmatrix} 0 & 0 & 1 \\ -2x & 2x & a-x \\ 0 & -2x & a+x \end{vmatrix} = 0$$ Expanding along R_1,

$$\Rightarrow (3a-x)\left[0-0+1\left(4x^2-0\right)\right]=0 \quad \Rightarrow x^2(3a-x)=0 \quad \therefore x = 0, 3a$$

Q13. (a) LHS : Let $\Delta = \begin{vmatrix} a+b & p+q & x+y \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$$ Taking 2 common from R_1,

$$\Rightarrow \Delta = 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$$ By $R_1 \to R_1 - R_2$

$$\Rightarrow \Delta = 2\begin{vmatrix} a & p & x \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix}$$ By $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta = 2\begin{vmatrix} a & p & x \\ b+c & q+r & y+z \\ c & r & z \end{vmatrix}$$ By $R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{RHS}$$

Alternatively, RHS : Let $\Delta = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$ By $R_1 \to R_1 + R_2,\ R_2 \to R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} a+b & p+q & x+y \\ b+c & q+r & y+z \\ 2c & 2r & 2z \end{vmatrix}$$ By $R_3 \to R_3 + R_1 - R_2$

$$\Rightarrow \Delta = \begin{vmatrix} a+b & p+q & x+y \\ b+c & q+r & y+z \\ c+a & r+p & z+x \end{vmatrix} = \text{LHS}$$

(b) LHS : Let $\Delta = \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$$ Taking 2 common from R_1,

$$\Rightarrow \Delta = 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$$ By $R_1 \to R_1 - R_2$

$$\Rightarrow \Delta = 2\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$$ By $R_3 \to R_3 - R_1$

$$\Rightarrow \Delta = 2\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ c & a & b \end{vmatrix}$$ By $R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{RHS}$$

Q14. Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 2(a+b+c) & 2(c+a+b) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$ Taking $2(a+b+c)$ common from R_1

$$\Rightarrow \Delta = 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$ By $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$\Rightarrow \Delta = 2(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ c-b & a-c & b+c \\ a-c & b-a & c+a \end{vmatrix}$$ Expanding along R_1,

$$\Rightarrow \Delta = 2(a+b+c)\left[0-0+1\left\{(c-b)(b-a)-(a-c)^2\right\}\right]$$

$$\Rightarrow \Delta = 2(a+b+c)\left\{bc-ac-b^2+ab+2ac-c^2-a^2\right\}$$

$$\Rightarrow \Delta = -2(a+b+c)\left\{a^2+b^2+c^2-ab-bc-ca\right\}$$

$$\therefore \Delta = 2(a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right) = \text{RHS.}$$

Q15. LHS : Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ By $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$ Taking $(a-b),(b-c)$ common from R_1 & R_2 respectively

$$\Rightarrow \Delta = (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$ Expanding along C_1,

$$\Rightarrow \Delta = (a-b)(b-c)\left[0-0+1(b+c-a-b)\right]$$

$$\Rightarrow \Delta = (a-b)(b-c)(c-a) = \text{RHS}$$

Note that this determinant is called a **circular determinant** and its value can be directly used in solving problems of objective types!

Q16. LHS : Let $\Delta = \begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$ By $R_1 \to aR_1,\ R_2 \to bR_2,\ R_3 \to cR_3$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ ba^2+abc & -abc & bc^2+abc \\ ca^2+abc & cb^2+abc & -abc \end{vmatrix}$$ Take a, b, c common from C_1, C_2 & C_3 resp.

$$\Rightarrow \Delta = \frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ba+bc & -ac & bc+ab \\ ca+bc & cb+ac & -ab \end{vmatrix}$$ By $R_1 \to R_1 + R_2,\ R_2 \to R_2 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} ba & ab & ac+2ab+bc \\ ba+2bc+ca & bc & bc \\ ca+bc & cb+ac & -ab \end{vmatrix}$$ By $R_1 \to R_1 + R_3$

$$\Rightarrow \Delta = \begin{vmatrix} ab+bc+ca & ab+bc+ca & ab+bc+ca \\ ba+2bc+ca & bc & bc \\ ca+bc & cb+ac & -ab \end{vmatrix}$$ Take $(ab+bc+ca)$ common from R_1,

$$\Rightarrow \Delta = (ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ba+2bc+ca & bc & bc \\ ca+bc & cb+ac & -ab \end{vmatrix} \quad \text{By } C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$$

$$\Rightarrow \Delta = (ab+bc+ca)\begin{vmatrix} 0 & 0 & 1 \\ ba+bc+ca & 0 & bc \\ 0 & cb+ac+ab & -ab \end{vmatrix} \quad \text{Expanding along } R_1,$$

$$\Rightarrow \Delta = (ab+bc+ca)\left[0+0+1\left\{(ab+bc+ca)^2 - 0\right\}\right]$$

$$\therefore \Delta = (ab+bc+ca)^3 = \text{RHS}$$

Q17. LHS : Let $\Delta = \begin{vmatrix} \sqrt{3}+\sqrt{5} & 2\sqrt{7} & \sqrt{7} \\ \sqrt{35}+\sqrt{6} & 7 & \sqrt{14} \\ 5+\sqrt{21} & \sqrt{35} & 7 \end{vmatrix} \Rightarrow \Delta = \begin{vmatrix} \sqrt{3} & 2\sqrt{7} & \sqrt{7} \\ \sqrt{6} & 7 & \sqrt{14} \\ \sqrt{21} & \sqrt{35} & 7 \end{vmatrix} + \begin{vmatrix} \sqrt{5} & 2\sqrt{7} & \sqrt{7} \\ \sqrt{35} & 7 & \sqrt{14} \\ 5 & \sqrt{35} & 7 \end{vmatrix}$

Taking $\sqrt{3}, \sqrt{7}$ and $\sqrt{7}$ common from C_1, C_2 and C_3 respectively in det(I).
Also taking $\sqrt{5}$ common from C_1 and, $\sqrt{7}$ from C_2 and C_3 both in det(II).

$$\therefore \Delta = 7\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{2} & \sqrt{7} & \sqrt{2} \\ \sqrt{7} & \sqrt{5} & \sqrt{7} \end{vmatrix} + 7\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{7} & \sqrt{7} & \sqrt{2} \\ \sqrt{5} & \sqrt{5} & \sqrt{7} \end{vmatrix} \quad \because C_1 \text{ and } C_2 \text{ are identical in } \det(I)$$

$$\Rightarrow \Delta = 7\sqrt{3} \times 0 + 7\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{7} & \sqrt{7} & \sqrt{2} \\ \sqrt{5} & \sqrt{5} & \sqrt{7} \end{vmatrix} \quad \text{By } C_1 \to C_1 - C_2$$

$$\Rightarrow \Delta = 7\sqrt{5}\begin{vmatrix} -1 & 2 & 1 \\ 0 & \sqrt{7} & \sqrt{2} \\ 0 & \sqrt{5} & \sqrt{7} \end{vmatrix} \quad \text{Expanding along } C_1$$

$$\Rightarrow \Delta = 7\sqrt{5}\left\{-1\left(7-\sqrt{10}\right) - 0 + 0\right\} \quad \therefore \Delta = 7\sqrt{5}\ [\sqrt{10} - 7] = \text{RHS}$$

Q18. We have $\begin{vmatrix} \alpha - x & \gamma & \beta \\ \gamma & \beta - x & \alpha \\ \beta & \alpha & \gamma - x \end{vmatrix} = 0$ By $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} \alpha+\beta+\gamma-x & \alpha+\beta+\gamma-x & \alpha+\beta+\gamma-x \\ \gamma & \beta - x & \alpha \\ \beta & \alpha & \gamma - a \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0-x & 0-x & 0-x \\ \gamma & \beta - x & \alpha \\ \beta & \alpha & \gamma - x \end{vmatrix} = 0 \quad [\text{Given } \alpha+\beta+\gamma = 0$$

Taking '$-x$' common from R_1, $-x\begin{vmatrix} 1 & 1 & 1 \\ \gamma & \beta - x & \alpha \\ \beta & \alpha & \gamma - x \end{vmatrix} = 0$

By $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$, $x\begin{vmatrix} 0 & 0 & 1 \\ \gamma-\beta+x & \beta-x-\alpha & \alpha \\ \beta-\alpha & \alpha-\gamma+x & \gamma-x \end{vmatrix} = 0$ Expanding along R_1,

$\Rightarrow x\left\{0-0+1\left[\gamma\alpha-\gamma^2+\gamma x-\beta\alpha+\beta\gamma-\beta x+x\alpha-x\gamma+x^2-\beta^2+\beta x+\beta\alpha+\alpha\beta-\alpha x-\alpha^2\right]\right\}=0$

$\Rightarrow x\left[x^2-\alpha^2-\beta^2-\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha\right]=0$

$\therefore x=0$ or $x^2=\alpha^2+\alpha^2+\beta^2+\gamma^2-[\alpha\beta+\beta\gamma+\gamma\alpha]$

$\Rightarrow x^2=\left(\alpha^2+\beta^2+\gamma^2\right)-\frac{1}{2}\left[(\alpha+\beta+\gamma)^2-\left(\alpha^2+\beta^2+\gamma^2\right)\right]$

$\Rightarrow x^2=\left(\alpha^2+\beta^2+\gamma^2\right)-\frac{1}{2}\left[0^2-\left(\alpha^2+\beta^2+\gamma^2\right)\right]$ $\Rightarrow x^2=\left(\alpha^2+\beta^2+\gamma^2\right)+\frac{1}{2}\left(\alpha^2+\beta^2+\gamma^2\right)$

$\Rightarrow x^2=\left(\alpha^2+\beta^2+\gamma^2\right)\left[1+\frac{1}{2}\right]$ i.e., $x=\pm\sqrt{\frac{3[\alpha^2+\beta^2+\gamma^2]}{2}}$

Hence $x=0$ or, $x=\pm\sqrt{\frac{3[\alpha^2+\beta^2+\gamma^2]}{2}}$.

Alternatively, we have got $x\left[x^2-\alpha^2-\beta^2-\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha\right]=0$

$\therefore x=0$ or $x^2=\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha$

$\Rightarrow x^2=\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\gamma\alpha-3(\alpha\beta+\beta\gamma+\gamma\alpha)$

$\Rightarrow x^2=(\alpha+\beta+\gamma)^2-\frac{3}{2}\left[2\alpha\beta+2\beta\gamma+2\gamma\alpha+\alpha^2+\beta^2+\gamma^2-(\alpha^2+\beta^2+\gamma^2)\right]$

$\Rightarrow x^2=(0)^2-\frac{3}{2}\left[(\alpha+\beta+\gamma)^2-(\alpha^2+\beta^2+\gamma^2)\right]$ i.e., $x^2=\frac{3}{2}\left[\alpha^2+\beta^2+\gamma^2\right]$

Hence $x=0$ or, $x=\pm\sqrt{\frac{3[\alpha^2+\beta^2+\gamma^2]}{2}}$.

Note that in the other sum : "If $a+b+c=0$, solve : $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0$." the symbols α, β and γ have been replaced by a, b and c respectively. So, it can also be solved in the same way as above.

Q19. LHS : Let $\Delta=\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$ $\Rightarrow \Delta=\begin{vmatrix} b^2c^2 & bc & c \\ c^2a^2 & ca & a \\ a^2b^2 & ab & b \end{vmatrix}+\begin{vmatrix} b^2c^2 & bc & b \\ c^2a^2 & ca & c \\ a^2b^2 & ab & a \end{vmatrix}$

Take c,a,b common from R_1,R_2,R_3 respectively in Δ_1

And b,c,a common from R_1,R_2,R_3 respectively in Δ_2

$\Rightarrow \Delta=abc\begin{vmatrix} b^2c & b & 1 \\ c^2a & c & 1 \\ a^2b & a & 1 \end{vmatrix}+abc\begin{vmatrix} bc^2 & c & 1 \\ ca^2 & a & 1 \\ ab^2 & b & 1 \end{vmatrix}$ [By $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$ in Δ_1 also, $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$ in Δ_2

$\Rightarrow \Delta=\frac{abc}{abc}\begin{vmatrix} ab^2c & ab & a \\ c^2ab & bc & b \\ a^2bc & ac & c \end{vmatrix}+\frac{abc}{abc}\begin{vmatrix} abc^2 & ca & a \\ bca^2 & ab & b \\ ab^2c & bc & c \end{vmatrix}$ [Taking abc common from C_1 in Δ_1 also, taking abc common from C_1 in Δ_2

$$\Rightarrow \Delta = abc\begin{vmatrix} b & ab & a \\ c & bc & b \\ a & ac & c \end{vmatrix} + abc\begin{vmatrix} c & ca & a \\ a & ab & b \\ b & bc & c \end{vmatrix}$$ $\left[\begin{array}{l}\text{By } R_1 \to cR_1,\ R_2 \to aR_2, R_3 \to bR_3 \text{ in } \Delta_1 \\ \text{also, } R_1 \to bR_1,\ R_2 \to cR_2, R_3 \to aR_3 \text{ in } \Delta_2\end{array}\right.$

$$\Rightarrow \Delta = \frac{abc}{abc}\begin{vmatrix} bc & abc & ac \\ ac & abc & ab \\ ab & abc & bc \end{vmatrix} + \frac{abc}{abc}\begin{vmatrix} bc & cab & ab \\ ac & abc & bc \\ ba & bca & ca \end{vmatrix}$$ $\left[\begin{array}{l}\text{Taking abc common from } C_2 \text{ in } \Delta_1 \\ \text{also, taking abc common from } C_2 \text{ in } \Delta_2\end{array}\right.$

$$\Rightarrow \Delta = abc\begin{vmatrix} bc & 1 & ac \\ ac & 1 & ab \\ ab & 1 & bc \end{vmatrix} + abc\begin{vmatrix} bc & 1 & ab \\ ac & 1 & bc \\ ba & 1 & ca \end{vmatrix} \qquad \Rightarrow \Delta = abc\left\{\begin{vmatrix} bc & 1 & ac \\ ac & 1 & ab \\ ab & 1 & bc \end{vmatrix} + \begin{vmatrix} bc & 1 & ab \\ ac & 1 & bc \\ ba & 1 & ca \end{vmatrix}\right\}$$

$$\Rightarrow \Delta = abc\begin{vmatrix} bc & 1 & ac+ab \\ ac & 1 & ab+bc \\ ab & 1 & bc+ca \end{vmatrix}$$ By $C_3 \to C_3 + C_1$

$$\Rightarrow \Delta = abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ac & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix}$$ $\because C_2$ and C_3 are proportional, $\therefore \Delta = abc \times 0 = 0$

Q20. Let A be the 1st term and R be the common ratio of given G.P.

$\therefore AR^9 = x,\ AR^{12} = y,\ AR^{14} = z$ $\quad \left[\because n^{th} \text{ term of a G.P., } A_n = AR^{n-1}\right.$

$$\text{Now } \det(A) = \begin{vmatrix} \log x & 10 & 1 \\ \log 4 & 13 & 1 \\ \log z & 15 & 1 \end{vmatrix} = \begin{vmatrix} \log AR^9 & 10 & 1 \\ \log AR^{12} & 13 & 1 \\ \log AR^{14} & 15 & 1 \end{vmatrix} \Rightarrow \quad = \begin{vmatrix} \log A + 9\log R & 10 & 1 \\ \log A + 12 \text{Log} R & 13 & 1 \\ \log A + 14\log R & 15 & 1 \end{vmatrix}$$

$$\Rightarrow \quad = \begin{vmatrix} \log A & 10 & 1 \\ \log A & 13 & 1 \\ \log A & 15 & 1 \end{vmatrix} + \begin{vmatrix} 9\log R & 10 & 1 \\ 12\log R & 13 & 1 \\ 14\log R & 15 & 1 \end{vmatrix}$$

Since C_1 and C_2 are proportional in det (I), and taking $\log R$ common from C_1 in det (II)

$$\Rightarrow \quad = 0 + \log R\begin{vmatrix} 9 & 10 & 1 \\ 12 & 13 & 1 \\ 14 & 15 & 1 \end{vmatrix}$$ By $C_1 \to C_1 + C_3$

$$\Rightarrow \quad = \log R\begin{vmatrix} 10 & 10 & 1 \\ 13 & 13 & 1 \\ 15 & 15 & 1 \end{vmatrix}$$ Since C_1 & C_2 are identical, $\therefore \det(A) = 0$

Q21. Given Given $-1 \le x < 0,\ 0 \le y < 1,\ 1 \le z < 2 ...(i)$

$$\text{Let } \Delta = \begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$$ By (i), $\Delta = \begin{bmatrix} -1+1 & 0 & 1 \\ -1 & 0+1 & 1 \\ -1 & 0 & 1+1 \end{bmatrix}$

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 0 - 0 + 1(0+1) = 1$$ [On expanding along R_1.

Q22. Let $\Delta = \begin{vmatrix} 1 & ab & 1/a+1/b \\ 1 & bc & 1/b+1/c \\ 1 & ca & 1/c+1/a \end{vmatrix}$ $\qquad R_1 \to R_1(c), R_2 \to R_2(a), R_3 \to R_3(b)$

$\Rightarrow \Delta = \dfrac{1}{abc}\begin{vmatrix} c & abc & \frac{c}{a}+\frac{c}{b} \\ a & abc & \frac{a}{b}+\frac{a}{c} \\ b & abc & \frac{b}{c}+\frac{b}{a} \end{vmatrix}$ $\qquad$ Taking abc common from C_2

$\Rightarrow \Delta = \dfrac{abc}{abc}\begin{vmatrix} c & 1 & \frac{c}{a}+\frac{c}{b} \\ a & 1 & \frac{a}{b}+\frac{a}{c} \\ b & 1 & \frac{b}{c}+\frac{b}{a} \end{vmatrix}$

Taking c, a, b common from R_1, R_2, R_3 respectively

$\Rightarrow \Delta = abc\begin{vmatrix} 1 & \frac{1}{c} & \frac{1}{a}+\frac{1}{b} \\ 1 & \frac{1}{a} & \frac{1}{b}+\frac{1}{c} \\ 1 & \frac{1}{b} & \frac{1}{c}+\frac{1}{a} \end{vmatrix}$ $\qquad$ By $C_3 \to C_3 + C_2 + C_1$

$\Rightarrow \Delta = abc\begin{vmatrix} 1 & \frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ 1 & \frac{1}{a} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ 1 & \frac{1}{b} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \end{vmatrix}$ $\qquad \because C_1$ and C_3 are proportional, $\Delta = 0$

Q23. As $\begin{vmatrix} x & -3i & 1 \\ y & 1 & i \\ 0 & 2i & -i \end{vmatrix} = 6+11i$ $\qquad \Rightarrow x\left(-i-2i^2\right)+3i\left(-iy-0\right)+1\left(2iy-0\right)=6+11i$

$\Rightarrow -xi+2x+3y+2iy = 6+11i$ $\qquad [\because i^2=-1$

$\Rightarrow 2x+3y+i(-x+2y)=6+11i$

By equating real and imaginary parts of the complex nos. on both sides, we get :

$2x+3y=6$, $-x+2y=11$ $\qquad \therefore x=-3,\ y=4$.

Q24. We have $f(x)=\begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$ $\qquad$ By $C_1 \to C_1 - xC_2$

$\Rightarrow f(x)=\begin{vmatrix} a+x & -1 & 0 \\ 0 & a & -1 \\ 0 & ax & a \end{vmatrix}$ $\qquad$ Expanding along C_1

$\Rightarrow f(x)=(a+x)(a^2+ax)-0+0$ $\qquad \therefore f(x)=a(a+x)^2$

Now $f(2x)-f(x)=a(a+2x)^2-a(a+x)^2$

$\Rightarrow f(2x)-f(x)=a[a^2+4x^2+4ax-a^2-x^2-2ax]$

$\Rightarrow f(2x)-f(x)=a[3x^2+2ax]$ $\quad\therefore f(2x)-f(x)=ax(3x+2a)$.

Q25. LHS: Let $\Delta=\begin{vmatrix}0&2&-3\\-2&0&4\\3&-4&0\end{vmatrix}$ Interchanging Rows & Columns,

$\Rightarrow \quad =\begin{vmatrix}0&-2&3\\2&0&-4\\-3&4&0\end{vmatrix}$ $\quad\Rightarrow \quad =(-1)(-1)(-1)\begin{vmatrix}0&2&-3\\-2&0&4\\3&-4&0\end{vmatrix}$

$\Rightarrow \Delta=-\Delta \quad \Rightarrow \Delta+\Delta=0 \quad \therefore \Delta=0=\text{RHS}$.

Q26. Let $\Delta=\begin{vmatrix}px+y&x&y\\py+z&y&z\\0&px+y&py+z\end{vmatrix}$

By $C_1\to C_1-pC_2-C_3$, $\Delta=\begin{vmatrix}0&x&y\\0&y&z\\-p^2x-py-py-z&px+y&py+z\end{vmatrix}$

Expanding along C_1, $\Delta=(-p^2x-2py-z)(xz-y^2)$

$\because$ x,y,z are in GP, so $xz=y^2\Rightarrow xz-y^2=0$ $\quad\therefore \Delta=0$.

Q27. We have $\begin{vmatrix}x+2&x+6&x-1\\x+6&x-1&x+2\\x-1&x+2&x+6\end{vmatrix}=0$ By $C_1\to C_1+C_2+C_3$

$\Rightarrow\begin{vmatrix}3x+7&x+6&x-1\\3x+7&x-1&x+2\\3x+7&x+2&x+6\end{vmatrix}=0$ By $R_1\to R_1-R_2, R_3\to R_3-R_1$

$\Rightarrow\begin{vmatrix}3x+7&x+6&x-1\\0&-7&3\\0&-4&7\end{vmatrix}=0$ On expanding along first column,

$\Rightarrow (3x+7)(-49+12)=0 \quad \therefore x=-\dfrac{7}{3}$.

Q28. Let $\Delta=\begin{vmatrix}yz-x^2&zx-y^2&xy-z^2\\zx-y^2&xy-z^2&yz-x^2\\xy-z^2&yz-x^2&zx-y^2\end{vmatrix}$ (By $c_1\to c_1-c_2$, $c_2\to c_2-c_3$

$\Rightarrow \quad =\begin{vmatrix}yz-zx-x^2+y^2&zx-xy-y^2+z^2&xy-z^2\\zx-xy-y^2+z^2&xy-yz-z^2+x^2&yz-x^2\\xy-yz-z^2+x^2&yz-zx-x^2+y^2&zx-y^2\end{vmatrix}$

$\Rightarrow \quad =\begin{vmatrix}(y-x)(x+y+z)&(z-y)(x+y+z)&xy-z^2\\(z-y)(x+y+z)&(x-z)(x+y+z)&yz-x^2\\(x-z)(x+y+z)&(y-x)(x+y+z)&zx-y^2\end{vmatrix}$

(Taking $(x+y+z)$ common from C_1 & C_2 both

$$\Rightarrow \quad =(x+y+z)^2\begin{vmatrix}(y-x) & (z-y) & xy-z^2\\(z-y) & (x-z) & yz-x^2\\(x-z) & (y-x) & zx-y^2\end{vmatrix} \qquad \text{(By } R_1 \to R_1+R_2+R_3$$

$$\Rightarrow \quad =(x+y+z)^2\begin{vmatrix}0 & 0 & xy+yz+zx-x^2-y^2-z^2\\(z-y) & (x-z) & yz-x^2\\(x-z) & (y-x) & zx-y^2\end{vmatrix} \qquad \text{(Expanding along } R_1$$

$$\Rightarrow \quad =(x+y+z)^2\left\{\left(xy+yz+zx-x^2-y^2-z^2\right)\left[(z-y)(y-x)-(x-z)^2\right]\right\}$$

$$\Rightarrow \quad =(x+y+z)^2\left\{\left(xy+yz+zx-x^2-y^2-z^2\right)\left(xy+yz+zx-x^2-y^2-z^2\right)\right\}$$

$$\Rightarrow \quad =(x+y+z)^2\left(x^2+y^2+z^2-xy-yz-zx\right)^2$$

$$\Rightarrow \quad =\frac{1}{4}(x+y+z)^2\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2$$

$$\Rightarrow \quad \Delta=(x+y+z)\left\{\frac{1}{4}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2\right\}$$

Hence, Δ is divisible by $(x+y+z)$ with a quotient of

$\frac{1}{4}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]^2$.

Q29. We have $\Delta=\begin{vmatrix}1 & 1 & 1\\1+\cos A & 1+\cos B & 1+\cos C\\\cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C\end{vmatrix}=0$ (By $c_1 \to c_1-c_2,\ c_2 \to c_2-c_3$

$$\Rightarrow\begin{vmatrix}0 & 0 & 1\\\cos A-\cos B & \cos B-\cos C & 1+\cos C\\\cos^2 A-\cos^2 B+\cos A-\cos B & \cos^2 B-\cos^2 C+\cos B-\cos C & \cos^2 C+\cos C\end{vmatrix}=0$$

(Taking $(\cos A-\cos B),(\cos B-\cos C)$ common from c_1, c_2 respectively

$$\Rightarrow(\cos A-\cos B)(\cos B-\cos C)\begin{vmatrix}0 & 0 & 1\\1 & 1 & 1\\\cos A+\cos B+1 & \cos B+\cos C+1 & \cos^2 C+\cos C\end{vmatrix}=0$$

Expanding along R_1, we get :

$(\cos A-\cos B)(\cos B-\cos C)\{0-0+1(\cos B+\cos C+1-\cos A-\cos B-1)\}=0$

$\Rightarrow(\cos A-\cos B)(\cos B-\cos C)(\cos C-\cos A)=0$

$\therefore$ Either $\cos A-\cos B=0 \Rightarrow \cos A=\cos B \Rightarrow A=B$ or, $\cos B-\cos C=0 \Rightarrow B=C$
or, $\cos C-\cos A=0 \Rightarrow C=A$

$\therefore \Delta ABC$ is an isosceles triangle if $\Delta=0$.

Q30. **LHS :** Let $\Delta=\begin{vmatrix}\frac{(a+b)^2}{c} & c & c\\a & \frac{(b+c)^2}{a} & a\\b & b & \frac{(c+a)^2}{b}\end{vmatrix}$

By $R_1 \to cR_1,\ R_2 \to aR_2,\ R_3 \to bR_3$,

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$$

By $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$,

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} (a+b+c)(a+b-c) & 0 & c^2 \\ 0 & (b+c+a)(b+c-a) & a^2 \\ (b+c+a)(b-c-a) & (b+c+a)(b-c-a) & (c+a)^2 \end{vmatrix}$$

Taking (a + b + c) common from C_1 and C_2 both

$$\Rightarrow \Delta = \frac{(a+b+c)^2}{abc}\begin{vmatrix} (a+b-c) & 0 & c^2 \\ 0 & (b+c-a) & a^2 \\ (b-c-a) & (b-c-a) & (c+a)^2 \end{vmatrix}$$

By $R_3 \to R_3 - (R_1 + R_2)$,

$$\Rightarrow \Delta = \frac{(a+b+c)^2}{abc}\begin{vmatrix} (a+b-c) & 0 & c^2 \\ 0 & (b+c-a) & a^2 \\ -2a & -2c & 2ca \end{vmatrix}$$

By $C_1 \to cC_1,\ C_2 \to aC_2$

$$\Rightarrow \Delta = \frac{(a+b+c)^2}{abcac}\begin{vmatrix} (ac+bc-c^2) & 0 & c^2 \\ 0 & (ba+ca-a^2) & a^2 \\ -2ca & -2ca & 2ca \end{vmatrix}$$

By $C_1 \to C_1 + C_3,\ C_2 \to C_2 + C_3$

$$\Rightarrow \Delta = \frac{(a+b+c)^2}{abcac}\begin{vmatrix} ac+bc & c^2 & c^2 \\ a^2 & ba+ca & a^2 \\ 0 & 0 & 2ca \end{vmatrix}$$

Take c, a, 2ca common from R_1, R_2 and R_3,

$$\Rightarrow \Delta = \frac{2c^2a^2(a+b+c)^2}{abcac}\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3,

$$\Rightarrow \Delta = \frac{2(a+b+c)^2}{b}\left[0 - 0 + 1\left(ab + ac + b^2 + bc - ac\right)\right]$$

$$\therefore \Delta = \frac{2(a+b+c)^2}{b}\left(ab + b^2 + bc\right) = 2(a+b+c)^2 = \text{RHS}.$$

Q31. We have $\begin{vmatrix} p & q & p\alpha + q \\ q & r & q\alpha + r \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = 0$

By $R_1 \to \alpha R_1$, $\frac{1}{\alpha}\begin{vmatrix} p\alpha & q\alpha & p\alpha^2+q\alpha \\ q & r & q\alpha+r \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix}=0$

By $R_3 \to R_3-(R_1+R_2)$, $\frac{1}{\alpha}\begin{vmatrix} p\alpha & q\alpha & p\alpha^2+q\alpha \\ q & r & q\alpha+r \\ 0 & 0 & -p\alpha^2-2q\alpha-r \end{vmatrix}=0$

Expanding along R_1, $\frac{1}{\alpha}\left[0-0+\left(-p\alpha^2-2q\alpha-r\right)\left(p\alpha r-q^2\alpha\right)\right]=0$

$\Rightarrow -\frac{1}{\alpha}\left(-\alpha(q^2-pr)\right)\left(2q\alpha+r+p\alpha^2\right)=0$

Either $\left(q^2-pr\right)=0$ or $\left(2q\alpha+r+p\alpha^2\right)=0$. That implies, $q^2=pr$ or $p(\alpha)^2+2q(\alpha)+r=0$

Therefore either p, q, r are in G.P. or, α satisfies the equation $px^2+2qx+r=0$.

Hence either p, q, r are in G.P. **or,** α is a root of the equation $px^2+2qx+r=0$.