Chapter 01

SOLUTIONS Of EXERCISE FOR PRACTICE

■ Based On Algebra Of Matrices

VERY SHORT ANSWER TYPE QUESTIONS

Q01. (a) We have $A=\begin{bmatrix}2&4\\3&2\end{bmatrix}$ and $B=\begin{bmatrix}-2&5\\3&4\end{bmatrix}$ $\therefore (3A-B)=3\begin{bmatrix}2&4\\3&2\end{bmatrix}-\begin{bmatrix}-2&5\\3&4\end{bmatrix}=\begin{bmatrix}8&7\\6&2\end{bmatrix}$.

(b) Given $A=\begin{bmatrix}1&3&2\end{bmatrix}$ and $B=\begin{bmatrix}6\\2\\3\end{bmatrix}$ $\therefore AB=\begin{bmatrix}1\ 3\ 2\end{bmatrix}_{1\times3}\begin{bmatrix}6\\2\\3\end{bmatrix}_{3\times1}=\begin{bmatrix}1\times6+3\times2+2\times3\end{bmatrix}=\begin{bmatrix}18\end{bmatrix}_{1\times1}$

Q02. (a) $A=\lambda\left[a_{ij}\right]_{3\times3}$ & $a_{ij}=\frac{2(9i-j)}{3}$ $\therefore a_{23}=\lambda\left[a_{23}\right]=\lambda\frac{2[9\times2-3]}{3}=10\lambda$

(b) Elements of $A=\left[a_{ij}\right]_{2\times2}$ are given by $a_{ij}=\frac{i}{j}$ $\therefore a_{12}=\frac{1}{2}$.

(c) We have $a_{ij}=\frac{|i-j|}{2}$ $\therefore a_{23}=\frac{|2-3|}{2}=\frac{1}{2}$.

(d) Since $a_{ij}=e^{2ix}\sin jx$ so, $a_{12}=e^{2x}\sin 2x$.

Q03. (a) Let A be the matrix of order 2×3 $\therefore A=\left[a_{ij}\right]_{2\times3}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$

Since there are 2 choices $(0$ or $1)$ to fill each places a_{ij} and repetition is allowed as well.

$\therefore$ Total no. of matrices $=2^6=64$.

(b) Let A be the matrix of order 3×3 $\therefore A=\left[a_{ij}\right]_{3\times3}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

Since we have 2 entries $(0$ and $1)$ to fill 9 places of a_{ij} and repetitions is allowed as well.

$\therefore$ Total no. of all possible matrices $=2^9=512$.

(c) No. of all possible matrices of order 2×2 with each entry 1, 2 or 3 is 3^4 or 81.

Q04. $A=\begin{bmatrix}3&0&0\\0&x&0\\0&0&3\end{bmatrix}$ $\because$ Matrix A is a scalar matrix $\therefore x=3$

Q05. Possible orders : $1\times12,\ 2\times6,\ 3\times4,\ 4\times3,\ 6\times2,\ 12\times1$

Q06. No. of rows in matrix $X=a+b$, No. of columns in Matrix $X=a+2$

No. of rows in matrix $Y=b+1$, No. of columns in matrix $Y=a+3$

Given that XY & YX both exist.

If XY exists, $a+2=b+1\Rightarrow a-b=-1\ldots$(i)

If YX exists, $a+b=a+3\Rightarrow b=3$

By (i), $a=-1+3=2$ $\therefore a=2$ and $b=3$.

Q07. (a) Order of $A=2\times3$, Order of $B=3\times5$

$\therefore$ Order of $AB=2\times5$. So, order of $(AB)^T=5\times2$.

(b) Order of $A=3\times4$, so order of $A^T=4\times3$. Let the order of $B=x\times y$.

If A^TB is defined, then $x=3$.

And if BA^T is defined, then $y=4$ $\therefore$ Order of $B=3\times4$.

Q08. $A=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ $\quad \therefore A^2 = A.A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I \quad [\because i^2 = -1$

Q09. a) Let A be the matrix of order 4×3, so $A=\left[a_{ij}\right]_{4\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$.

Given that $a_{ij} = \dfrac{i-j}{i+j}$

$\therefore a_{11} = \dfrac{1-1}{1+1} = 0,\ a_{12} = \dfrac{1-2}{1+2} = \dfrac{-1}{3},\ a_{13} = \dfrac{1-3}{1+3} = \dfrac{-1}{2},$

$a_{21} = \dfrac{2-1}{2+1} = \dfrac{1}{3},\ a_{22} = \dfrac{2-2}{2+3} = 0,\ a_{23} = \dfrac{2-3}{2+3} = \dfrac{-1}{5},$

$a_{31} = \dfrac{3-1}{3+1} = \dfrac{1}{2},\ a_{32} = \dfrac{3-2}{3+2} = \dfrac{1}{5},\ a_{33} = \dfrac{3-3}{3+3} = 0,$

$a_{41} = \dfrac{4-1}{4+1} = \dfrac{3}{5},\ a_{42} = \dfrac{4-2}{4+2} = \dfrac{1}{3},\ a_{43} = \dfrac{4-3}{4+3} = \dfrac{1}{7}$

$\therefore A = \begin{bmatrix} 0 & -1/3 & -1/2 \\ 1/3 & 0 & -1/5 \\ 1/2 & 1/5 & 0 \\ 3/5 & 1/3 & 1/7 \end{bmatrix}$

b) Let B be the matrix of order 3×2. Given that $[b_{ij}] = \dfrac{|i-2j|}{3}$.

Assume that, $B=\left[b_{ij}\right]_{3\times2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

$\therefore b_{11} = \dfrac{|1-2\times1|}{3} = \dfrac{1}{3},\ b_{12} = \dfrac{|1-2\times2|}{3} = 1,\ b_{21} = \dfrac{|2-2\times1|}{3} = 0,\ b_{22} = \dfrac{|2-2\times2|}{3} = \dfrac{2}{3}$

$b_{31} = \dfrac{|3-2\times1|}{3} = \dfrac{1}{3},\ b_{32} = \dfrac{|3-2\times2|}{3} = \dfrac{1}{3}$ $\qquad \therefore B = [bij]_{3\times2} = \begin{bmatrix} 1/3 & 1 \\ 0 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$

c) Let A be the matrix of order 2×3. $\therefore A = \left[a_{ij}\right]_{2\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$\therefore a_{11} = 1-1 = 0,\ a_{12} = -1+3(2) = 5,\ a_{13} = -1+3(3) = 8$

$a_{21} = 2-2(1) = 0,\ a_{22} = 2-2 = 0,\ a_{23} = -2+3(3) = 7$

$\therefore A = \left[a_{ij}\right]_{2\times3} = \begin{bmatrix} 0 & 5 & 8 \\ 0 & 0 & 7 \end{bmatrix}$

Q10. (a) $[x \ \ 1]_{1\times2} \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}_{2\times2} \begin{bmatrix} x \\ 5 \end{bmatrix} = O \quad \Rightarrow [(x-2) \quad -3]\begin{bmatrix} x \\ 5 \end{bmatrix} = O \quad \therefore \left[x^2-2x-15\right] = O$

$\Rightarrow \left[x^2-2x-15\right] = [0] \quad \Rightarrow x^2-2x-15=0 \quad \Rightarrow x^2-5x+3x-15=0$

$\Rightarrow (x+3)(x-5)=0 \quad \therefore x=-3 \ \& \ x=5$

(b) $\begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}1 & 5\end{bmatrix}=\begin{bmatrix}3 & 7x+y\\2y & 10\end{bmatrix} \quad \Rightarrow \begin{bmatrix}3 & 15\\2 & 10\end{bmatrix}=\begin{bmatrix}3 & 7x+y\\2y & 10\end{bmatrix}$

By equality of matrices, $2y=2\Rightarrow y=1$, $7x+y=15\Rightarrow 7x=14\Rightarrow x=2 \quad \therefore x=2\ \&\ y=1$.

(c) Given $[x \quad 1]\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix}=O \quad \Rightarrow [x-2 \quad 0]=[0\ \ 0]$

By equality of matrices, $x-2=0 \quad \therefore x=2$.

(d) Do yourself. Value of x : –1. (e) Do yourself. Value of x : $\pm 4\sqrt{3}$.

Q11. (a) Given $2A-B=\begin{bmatrix}4 & -6\\-4 & 2\end{bmatrix}...(i)$, $A+2B=\begin{bmatrix}-1 & 0\\1 & 1\end{bmatrix}...(ii)$

By 2(i) + (ii), $2(2A-B)+(A+2B)=\begin{bmatrix}8 & -10\\-8 & 4\end{bmatrix}+\begin{bmatrix}-1 & 0\\1 & 1\end{bmatrix} \quad \Rightarrow 5A=\begin{bmatrix}7 & -12\\-7 & 5\end{bmatrix}$

$\Rightarrow A=\begin{bmatrix}7/5 & -12/5\\-7/5 & 1\end{bmatrix}$.

Also by (i) – 2 (ii), $(2A-B)-2(A+2B)=\begin{bmatrix}4 & -6\\-4 & 2\end{bmatrix}-\begin{bmatrix}-2 & 0\\2 & 2\end{bmatrix} \quad \Rightarrow -5B=\begin{bmatrix}6 & -6\\-6 & 0\end{bmatrix}$

$\Rightarrow B=\begin{bmatrix}-6/5 & 6/5\\6/5 & 0\end{bmatrix}$

(b) Given $2X+3Y=\begin{bmatrix}2 & 3\\4 & 0\end{bmatrix}...(i)$, $3X+2Y=\begin{bmatrix}2 & -2\\-1 & 5\end{bmatrix}...(ii)$

By $2(i)-3(ii)$, $4X+6Y-9X-6Y=\begin{bmatrix}4 & 6\\8 & 0\end{bmatrix}-\begin{bmatrix}6 & -6\\-3 & 15\end{bmatrix} \quad \Rightarrow -5X=\begin{bmatrix}-2 & 12\\11 & -15\end{bmatrix}$

$\Rightarrow X=\begin{bmatrix}2/5 & -12/5\\-11/5 & 3\end{bmatrix}$

Also by $3(i)-2(ii)$, $6X+9Y-6X-4Y=\begin{bmatrix}6 & 9\\12 & 0\end{bmatrix}-\begin{bmatrix}4 & -4\\-2 & 10\end{bmatrix} \quad \Rightarrow 5Y=\begin{bmatrix}2 & 13\\14 & -10\end{bmatrix}$

$\Rightarrow Y=\begin{bmatrix}2/5 & 13/5\\14/5 & -2\end{bmatrix}$

Q12. $A=\text{diag}[1 \quad -1 \quad 2]=\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 2\end{bmatrix}$ and, $B=\text{diag}[2 \quad 3 \quad -1]=\begin{bmatrix}2 & 0 & 0\\0 & 3 & 0\\0 & 0 & -1\end{bmatrix}$

$\therefore 3A+4B=\begin{bmatrix}3 & 0 & 0\\0 & -3 & 0\\0 & 0 & 6\end{bmatrix}+\begin{bmatrix}8 & 0 & 0\\0 & 12 & 0\\0 & 0 & -4\end{bmatrix} \quad \Rightarrow 3A+4B=\begin{bmatrix}11 & 0 & 0\\0 & 9 & 0\\0 & 0 & 2\end{bmatrix}$

Therefore, $3A+4B=\text{diag}[11 \quad 9 \quad 2]$

Q13. $A=\begin{bmatrix}8 & 0\\4 & -2\\3 & 6\end{bmatrix}, B=\begin{bmatrix}2 & -2\\4 & 2\\-5 & 1\end{bmatrix}$ and $2A+3X=5B$

$$\therefore \begin{bmatrix} 16 & 0 \\ 8 & -4 \\ 6 & 12 \end{bmatrix} + 3X = \begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} \quad \Rightarrow 3X = \begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} \quad \text{So, } X = \begin{bmatrix} -2 & -10/3 \\ 4 & 14/3 \\ -31/3 & -7/3 \end{bmatrix}$$

Q14. According to question, $A = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now by equality of matrices, $\cos\omega = 1,\ \sin\omega = 0$ $\quad \Rightarrow \cos\omega = \cos 0,\ \sin\omega = \sin 0$

$\therefore \omega = 2n\pi,\ n \in Z$ and, $\omega = n\pi,\ n \in Z$ $\quad$ So, $\omega = 2n\pi,\ n \in Z$

Q15. $A = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ $\quad$ Given that $A + A^T = I_2$

$\therefore \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} + \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \Rightarrow \begin{bmatrix} 2\cos\vartheta & 0 \\ 0 & 2\cos\vartheta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By equality of matrices, $2\cos\vartheta = 1$ $\quad \Rightarrow \cos\vartheta = 1/2$

$\Rightarrow \cos\vartheta = \cos\left(\frac{\pi}{3}\right)$ $\quad \therefore \vartheta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

Q16. $A = \begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$ $\quad \Rightarrow A' = \begin{bmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{bmatrix}$

LHS: $A'A = \begin{bmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{bmatrix} \begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$

$$\Rightarrow \quad = \begin{bmatrix} \sin^2 x + \cos^2 x & \sin x \cos x - \sin x \cos x \\ \sin x \cos x - \sin x \cos x & \sin^2 x + \cos^2 x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = RHS$$

Q17. $\cos\omega \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} + \sin\omega \begin{bmatrix} \sin\omega & -\cos\omega \\ \cos\omega & \sin\omega \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\omega & \cos\omega\sin\omega \\ -\cos\omega\sin\omega & \cos^2\omega \end{bmatrix} + \begin{bmatrix} \sin^2\omega & -\sin\omega\cos\omega \\ \sin\omega\cos\omega & \sin^2\omega \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\omega + \sin^2\omega & \cos\omega\sin\omega - \cos\omega\sin\omega \\ -\cos\omega\sin\omega + \cos\omega\sin\omega & \cos^2\omega + \sin^2\omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Q18. $A = \begin{bmatrix} \omega & \kappa \\ \eta & -\omega \end{bmatrix}$ and, $A^2 = I$

$\therefore A^2 = A.A = \begin{bmatrix} \omega & \kappa \\ \eta & -\omega \end{bmatrix} . \begin{bmatrix} \omega & \kappa \\ \eta & -\omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \Rightarrow \begin{bmatrix} \omega^2 + \kappa\eta & \omega\kappa - \omega\kappa \\ \omega\eta - \omega\eta & \eta\kappa + \omega^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \omega^2 + \kappa\eta & 0 \\ 0 & \omega^2 + \kappa\eta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ By equality of matrices, we have :

$\omega^2 + \kappa\eta = 1$ and $\omega^2 + \kappa\eta = 1$ $\quad \therefore 1 - \omega^2 - k\eta = 0$

Q19. **(a)** $A^2 = A \ldots(i)$

Let $P = (I + A)^3 - 7A = (I + A)(I + A)(I + A) - 7A$

$\Rightarrow \quad = (I.I + I.A + A.I + A.A)(I + A) - 7A = (I + 2A + A^2)(I + A) - 7A$

$\Rightarrow \quad = (I + 3A)(I + A) - 7A$ $\quad$ [By (i)]

$\Rightarrow \quad = (I.I + I.A + 3A.I + 3A^2) - 7A = I + A + 3A + 3A - 7A$ $\quad \therefore \quad P = I.$

(b) Here $A^2 = I = A.A \ldots (i)$

Let $P = (A-I)^3 + (A+I)^3 - 7A = (A-I)(A-I)(A-I) + (A+I)(A+I)(A+I) - 7A$

$\Rightarrow \quad = (A.A - A.I - I.A + I.I)(A-I) + (A.A + A.I + I.A + I.I)(I+A) - 7A$

$\Rightarrow \quad = (A^2 - A - A + I)(A-I) + (A^2 + A + A + I)(I+A) - 7A$

$\Rightarrow \quad = (I - 2A + I)(A-I) + (I + 2A + I)(I+A) - 7A$ [By (i)

$\Rightarrow \quad = 2(-A + I)(A-I) + 2(A+I)(I+A) - 7A$

$\Rightarrow \quad = 2(-A.A + A.I + I.A - I.I) + 2(A.I + A.A + I.I + I.A) - 7A$

$\Rightarrow \quad = 2(-I + A + A - I) + 2(A + I + I + A) - 7A = 4(A-I) + 4(A+I) - 7A$

$\Rightarrow \quad = A \qquad \therefore \quad P = A$.

Q20. Please see in the theory section of this chapter in Mathematicia Vol. 1.

Q21. (a) $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

By equality of matrices, $a - b = -1 - (i)$, $2a + c = 8$, $2a - b = 0 - (ii)$, $3c + d = 13$.

By (i) and (ii), $a - 2a = -1 \Rightarrow a = 1$ & $b = 2$

(b) $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$

(c) Given $\begin{pmatrix} a+4 & 3b \\ 8 & -6 \end{pmatrix} = \begin{pmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{pmatrix}$

By equality of matrices, $a + 4 = 2a + 2,\ 3b = b + 2,\ -6 = a - 8b$

Solving these equations, we get : $a = 2,\ b = 1 \qquad \therefore a - 2b = 2 - 2(1) = 0$.

(d) $\begin{bmatrix} -2 & 4 & -2 \\ 3 & 7 & 3 \end{bmatrix} + A = \begin{bmatrix} -1 & 2 & 6 \\ 4 & 5 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -1 & 2 & 6 \\ 4 & 5 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 4 & -2 \\ 3 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 8 \\ 1 & -2 & -3 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix} - A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & -10 \end{bmatrix}$

Q22. **a)** $\begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix} = 2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix} = \begin{bmatrix} 2x+3 & 14 \\ 15 & 2y-4 \end{bmatrix}$

By equality of matrices, $2x + 3 = 7 \Rightarrow x = 2$ & $2y - 4 = 14 \Rightarrow y = 9 \qquad \therefore x = 2,\ y = 9$

b) $\begin{bmatrix} x+y & 3 \\ 7 & xy \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 7 & -12 \end{bmatrix}$

By equality of matrices, $x + y = 1 \ldots (i)$ & $xy = -12 \ldots (ii)$

By (i) & (ii), $x(1-x) = -12 \qquad \Rightarrow x - x^2 = -12 \qquad \Rightarrow x^2 - x - 12 = 0$

$\Rightarrow x^2 - 4x + 3x - 12 = 0 \qquad \Rightarrow (x+3)(x-4) = 0 \qquad \therefore x = -3$ & $x = 4$

If $x = -3$, then $y = 1 - x = 1 + 3 = 4$.

And, if $x = 4$ then, $y = 1 - x = 1 - 4 = -3$.

c) $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - \begin{bmatrix} 5x \\ 6y \end{bmatrix} = 3\begin{bmatrix} -2 \\ -3 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} x^2 - 5x \\ y^2 - 6y \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \end{bmatrix}$

By quality of matrices, $x^2 - 5x = -6$ & $y^2 - 6y = -9$

$\Rightarrow x^2-5x+6=0...(i)$, $y^2-6y+9=0\ ...(ii)$

By (i), $x^2-5x+6=0 \quad \Rightarrow x^2-3x-2x+6=0 \quad \Rightarrow (x-2)(x-3)=0 \qquad \therefore x=2,3$

By (ii), $y^2-6y+9=0 \quad \Rightarrow y^2-3y-3y+9=0 \quad \Rightarrow (y-3)(y-3)=0 \qquad \therefore y=3$

Therefore, $x=2,\ 3;\ y=3$.

d) $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - \begin{bmatrix} 3x \\ 6y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x^2-3x \\ y^2-6y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$

By equality of matrices,

$x^2-3x=-2,\ y^2-6y=9 \ \Rightarrow x^2-3x+2=0...(i),\ y^2-6y-9=0...(ii)$

By (i), $x^2-3x+2=0 \quad \Rightarrow x^2-2x-x+2=0 \quad \Rightarrow (x-1)(x-2)=0 \qquad \therefore x=1,\ 2$

By (ii), $y^2-6y-9=0 \quad \Rightarrow y=\dfrac{6\pm\sqrt{36+36}}{2}=\dfrac{6\pm\sqrt{72}}{2}=\dfrac{6\pm 6\sqrt{2}}{2} \qquad \therefore y=3\pm 3\sqrt{2}$

e) $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+a \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

By equality of matrices, $x-y=-1\ ...(i)$, $\quad 2x+z=5\ ...(iii)$

$2x-y=0\ ...(iii)$, $\quad 3z+a=13\ ...(iv)$

By (i) & (iii), $x-2x=-1 \Rightarrow x=1 \quad \therefore y=2$

Replacing value of x in (iii), $2(1)+z=5 \Rightarrow z=3$ also, $3(3)+a=13 \Rightarrow a=4$

$\therefore x=1,\ y=2,\ z=3,\ a=4$.

f) $\begin{bmatrix} 2x+y & 3y \\ 0 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$

By equality of matrices, $2x+y=x+3 \Rightarrow x+y=3\ ...(i)$,

$3y=y^2+2 \Rightarrow y^2-3y+2=0\ ...(ii)$

By (ii), $y^2-3y+2=0 \quad \Rightarrow y^2-2y-y+2=0 \quad \Rightarrow (y-1)(y-2)=0 \quad \Rightarrow y=1,\ 2$

If $y=1$, then $x+1=3 \Rightarrow x=2\ ...(A)$

If $y=2$, then $x+2=3 \Rightarrow x=1$

Note that y = 1 [from (A)] doesn't satisfy the matrix equation so, $x=1,\ y=2$

g) $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

By equality of matrices, $x+3=0 \Rightarrow x=-3,\ z+4=6 \Rightarrow z=2,\ 2y-7=3y-2 \Rightarrow y=-5$,

$a-1=-3 \Rightarrow a=-2,\ 3b=-21 \Rightarrow b=-7,\ 2c+2=0 \Rightarrow c=-1$

$\therefore a=-2,\ b=-7,\ c=-1,\ x=-3,\ y=-5,\ z=2$.

h) $A=\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix},\ B\begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}$

As $A^2+B^2=(A+B)^2 \quad \Rightarrow A^2+B^2=(A+B)(A+B)=A^2+AB+BA+B^2$

By Cancellation Laws, we get : $AB=-BA$

$\therefore \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix} = -\begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x-y & 2 \\ 2x-y & 3 \end{bmatrix} = \begin{bmatrix} -x-2 & x+1 \\ -y+2 & y-1 \end{bmatrix}$

By equality of matrices : $x+1=2 \Rightarrow x=1,\ y-1=3 \Rightarrow y=4 \qquad \therefore x=1,\ y=4$

i) $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} 4+x & x+y+6 \\ z+w-1 & 2w+3 \end{bmatrix}$

By equality of matrices, $3x = 4 + x \Rightarrow x = 2$, $3y = x + y + 6 \Rightarrow 2y = 2 + 6 \Rightarrow y = 4$,
$2w + 3 = 3w \Rightarrow w = 3$ & $3z = z + w - 1 \Rightarrow 2z = 3 - 1 \Rightarrow z = 1$ $\quad \therefore w = 3, x = 2, y = 4, z = 1$

j) $\begin{bmatrix} x & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = O \quad \Rightarrow \begin{bmatrix} 2x+2-2 & x+6-2 & -x-4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x & x+4 & -(x+4) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 2x + x + 4 - (x+4) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \quad \Rightarrow 2x = 0 \quad \therefore x = 0.$

Q23. Given $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix} = I_3$

$\Rightarrow \begin{bmatrix} -2x+7x & 28x-28x & 14x-14x \\ 0 & 1 & 0 \\ -x+x & 14x-2-4x & 7x-2x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

By equality of matrices : $5x = 1 \Rightarrow x = 1/5$, $10x - 2 = 0 \quad \Rightarrow x = 2/10 = 1/5$

Q24. Given that $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies $A^T A = I$

$\therefore \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = I \quad \Rightarrow \begin{bmatrix} x^2 + x^2 & xy - xy & -xz + xz \\ xy - xy & 4y^2 + y^2 + y^2 & 2yz - yz - yz \\ -zx + zx & 2yz - zy - zy & z^2 + z^2 + z^2 \end{bmatrix} = I$

$\Rightarrow \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By equality of matrices, we get : $2x^2 = 1 \Rightarrow x \pm \frac{1}{\sqrt{2}}$, $6y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{6}}$, $3z^2 = 1 \Rightarrow z \pm \frac{1}{\sqrt{3}}$.

Or, Here $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \quad \therefore A' = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$.

As $A^{-1} = A' \Rightarrow AA^{-1} = AA' \Rightarrow AA' = I \ldots$(i)

Now by (i), $\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By equality of matrices, $2x^2 = 1$, $6y^2 = 1$, $3z^2 = 1$ i.e., $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{6}}$, $z = \pm\frac{1}{\sqrt{3}}$.

Q25. Note that $(2 \quad 1 \quad 3)_{1\times3}\begin{pmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}_{3\times3}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}_{3\times1} = A$ so, order of matrix A is 1×1.

Q26. Let A be a square matrix.

Then, $A = \frac{A + A^T}{2} + \frac{A - A^T}{2} = P + Q$, where $P = \frac{A + A^T}{2}$, $Q = \frac{A - A^T}{2}$

Now $P = \frac{A + A^T}{2} \quad \Rightarrow P^T = \left(\frac{A + A^T}{2}\right)^T \quad \Rightarrow P^T = \frac{A^T + [A^T]^T}{2} = \frac{A^T + A}{2}$

$\therefore P = P^T$ So, P is symmetric matrix.

Also, $Q = \frac{A - A^T}{2} \quad \Rightarrow Q^T = \left(\frac{A - A^T}{2}\right)^T \quad \Rightarrow Q^T = \frac{A^T - [A^T]^T}{2} = \frac{A^T - A}{2} = -\frac{A - A^T}{2}$

$\therefore Q = -Q^T$ So, Q is skew-symmetric matrix.

Therefore, A = P + Q, where P is symmetric and Q is skew-symmetric matrix.
Hence A is expressible as the sum of a symmetric and a skew-symmetric matrix.

Q27. Given that A and B are both symmetric matrices $\therefore A = A^T \ \& \ B = B^T \ldots$(i)

Let $P = AB \quad \Rightarrow P^T = (AB)^T = B^T A^T \quad \Rightarrow P^T = BA$ [By (i)

If A and B commute then, $AB = BA \quad \therefore P^T = AB \quad \Rightarrow P^T = P$

So, P is symmetric matrix.

Q28. Given that A and B are symmetric matrix so, $A = A^T \ \& \ B = B^T \ldots$(i)

Let $P = AB - BA \quad \Rightarrow P^T = (AB - BA)^T = B^T A^T - A^T B^T$

$\Rightarrow P^T = BA - AB \quad \Rightarrow P^T = -P$ [By using (i)]

Therefore, P is a skew-symmetric matrix.

Q29. If A be a symmetric matrix then, $A = A^T$

Let $P = B^T AB \quad \Rightarrow P^T = B^T A^T (B^T)^T = B^T A^T B \quad \Rightarrow P^T = B^T AB = P$

So, P is symmetric matrix if A is symmetric.

If A is a skew-symmetric matrix then, $A = -A^T$

Then $P = B^T AB \quad \Rightarrow P^T = B^T A^T (B^T)^T = B^T A^T B \quad \Rightarrow P^T = B^T(-A)B$

$\Rightarrow P^T = -B^T AB \quad \Rightarrow P^T = -P$

So, P is skew-symmetric matrix if A is skew-symmetric matrix.

Q30. B is a skew-symmetric matrix, then $B = -B^T \ldots$(i)

Let $P = ABA^T \quad \Rightarrow P^T = (ABA^T)^T \quad \Rightarrow P^T = (A^T)^T B^T A^T \quad \Rightarrow P^T = A(-B)A^T$

$\Rightarrow P^T = -ABA^T \quad \Rightarrow P^T = -P$

$\therefore$ P is skew-symmetric matrix if B is skew-symmetric matrix.

Q31. **(a)** Since A is a skew-symmetric matrix so, $A = -A^T$

Now, $A = \begin{bmatrix} 0 & 5 & -3 \\ -5 & p & 4 \\ q & -4 & 0 \end{bmatrix} \qquad \therefore \begin{bmatrix} 0 & 5 & -3 \\ -5 & p & 4 \\ q & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5 & -q \\ -5 & -p & 4 \\ 3 & -4 & 0 \end{bmatrix}$

By equality of matrices, $q = 3,\ p = -p \Rightarrow p = 0 \qquad \therefore p = 0,\ q = 3$

Note that all the diagonal elements in a skew-symmetric matrix are 0 so, p = 0 could have been easily obtained.

(b) Method 1 : Since A is symmetric so, $A = A' \qquad \Rightarrow \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix}$

By equality of matrices, we get : $-2 = 3a,\ 2b = 3 \qquad \therefore a = -2/3,\ b = 3/2$.

Method 2 : Since A is symmetric so, $A = A'$ i.e., $a_{ij} = a_{ji}$ where $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$

$\therefore a_{12} = a_{21} \Rightarrow 2b = 3 \Rightarrow b = 3/2$ and, $a_{13} = a_{31} \Rightarrow -2 = 3a \Rightarrow b = -2/3$.

(c) $A = \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix} \quad \Rightarrow A^T = \begin{pmatrix} 3 & 7 \\ 5 & 9 \end{pmatrix}$

$\Rightarrow P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix} + \begin{pmatrix} 3 & 7 \\ 5 & 9 \end{pmatrix}\right\} = \begin{pmatrix} 3 & 6 \\ 6 & 9 \end{pmatrix}$.

Q32. Given $Z = \begin{Vmatrix} 10 & -2 \\ -5 & 1 \end{Vmatrix}$. As we know that $ZZ^{-1} = I$, so $\begin{Vmatrix} 10 & -2 \\ -5 & 1 \end{Vmatrix} Z^{-1} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$

By $R_1 \to R_1 + 2R_2$, $\begin{Vmatrix} 0 & 0 \\ -5 & 1 \end{Vmatrix} Z^{-1} = \begin{Vmatrix} 1 & 2 \\ 0 & 1 \end{Vmatrix}$ $\quad \because$ all elements in R_1 are 0 so, Z^{-1} doesn't exist.

Q33. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. As $A^4 = A^2.A^2 \ldots$(i)

Now, $A^2 = A.A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A^4 = A^2.A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.I \qquad \Rightarrow A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Q34. We have $A = \begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}$ $\qquad$ As $A^{20} = \left(A^2\right)^{10}$

Now, $A^2 = A.A = \begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}.\begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

$\therefore A^{20} = \left(A^2\right)^{10} = (O)^{10} = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Q35. (a) Given $2\begin{pmatrix} 3 & 4 \\ 5 & x \end{pmatrix} + \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 10 & 5 \end{pmatrix}$ $\qquad \Rightarrow \begin{pmatrix} 6 & 8 \\ 10 & 2x \end{pmatrix} + \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 10 & 5 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 7 & 8+y \\ 10 & 2x+1 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 10 & 5 \end{pmatrix}$ $\qquad$ By equality of matrices : $8 + y = 0,\ 2x + 1 = 5$

$\Rightarrow y = -8,\ x = 2$ $\qquad$ Therefore, $(x - y) = 2 - (-8) = 10$.

(b) Given that $A = \begin{pmatrix} 0 & 3 \\ 2 & -5 \end{pmatrix}$ and $kA = \begin{pmatrix} 0 & 4a \\ -8 & 5b \end{pmatrix}$

So, $k\begin{pmatrix} 0 & 3 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 4a \\ -8 & 5b \end{pmatrix}$ $\qquad \Rightarrow \begin{pmatrix} 0 & 3k \\ 2k & -5k \end{pmatrix} = \begin{pmatrix} 0 & 4a \\ -8 & 5b \end{pmatrix}$

By equality of matrices, $3k = 4a,\ 2k = -8,\ -5k = 5b$

Solving these equation simultaneously we get : $k = -4,\ a = -3$.

(c) Do yourself. Ans. $x + y = 2 + 9 = 11$

Q36. Given $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

Applying elementary column operations $C_2 \to C_2 - 2C_1$ in the matrix equation, we get :

$$\Rightarrow \begin{pmatrix} 4 & -6 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -1 \end{pmatrix}$$

Q37. Given $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} A = A^2$

$\Rightarrow \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} AA^{-1} = AAA^{-1}$ [Post-multiplying by A^{-1} both sides

$\Rightarrow \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} I = A I$ $\therefore A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$

Q38. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

SHORT & LONG ANSWER TYPE QUESTIONS

Q01. Given $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ $\Rightarrow A^T = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

Now $AA^T = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

$$\Rightarrow AA^T = \begin{bmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_2l_1+m_2m_1+n_2n_1 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_3l_1+m_3m_1+n_3n_1 & l_3l_2+m_3m_2+n_3n_2 & l_3^2+m_3^2+n_3^2 \end{bmatrix}$$

Since lines are perpendicular so, $AA^T = \begin{bmatrix} l_1^2+m_1^2+n_1^2 & 0 & 0 \\ 0 & l_2^2+m_2^2+n_2^2 & 0 \\ 0 & 0 & l_3^2+m_3^2+n_3^2 \end{bmatrix}$

$\Rightarrow AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$. Hence Proved.

Note that this question is based on the concepts of direction cosines of lines from Chapter 11 of NCERT Textbook Part II **Three Dimensional Geometry**.

Q02. (a) Given $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

Now $A + B = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -10 & 0 \end{bmatrix}$

$\therefore (A+B)C = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -10 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-14+24 \\ -10+0+30 \\ 16+20+0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 36 \end{bmatrix} \ldots(i)$

Also $AC=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-12+21\\-12+0+24\\14+16+0\end{bmatrix}=\begin{bmatrix}9\\12\\30\end{bmatrix}$

and, $BC=\begin{bmatrix}0&1&1\\1&0&2\\1&-2&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-2+3\\2+0+6\\2+4+0\end{bmatrix}=\begin{bmatrix}1\\8\\6\end{bmatrix}$ $\Rightarrow AC+BC=\begin{bmatrix}9\\12\\30\end{bmatrix}+\begin{bmatrix}1\\8\\6\end{bmatrix}=\begin{bmatrix}10\\20\\36\end{bmatrix}\ldots$(ii)

By (i) and (ii), it is clear that $(A+B)C=AC+BC$

☞ This property is known as the **distributive property of matrix addition**.

(b) We have $AC=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-12+21\\-12+0+24\\14+16+0\end{bmatrix}=\begin{bmatrix}9\\12\\30\end{bmatrix}$,

And, $BC=\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-2+3\\2+0+6\\2-4+0\end{bmatrix}=\begin{bmatrix}1\\8\\-2\end{bmatrix}$.

Also $A+B=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}+\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix}=\begin{bmatrix}0&7&8\\-5&0&10\\8&-6&0\end{bmatrix}$

$\Rightarrow (A+B)C=\begin{bmatrix}0&7&8\\-5&0&10\\8&-6&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-14+24\\-10+0+30\\16+12+0\end{bmatrix}=\begin{bmatrix}10\\20\\28\end{bmatrix}\ldots$(i)

And, $AC+BC=\begin{bmatrix}9\\12\\30\end{bmatrix}+\begin{bmatrix}1\\8\\-2\end{bmatrix}=\begin{bmatrix}10\\20\\28\end{bmatrix}$...(ii)

By (i) & (ii), it is clear that (A + B) C = AC + BC.

Q03. Given $A\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}=\begin{bmatrix}-7&-8&-9\\2&4&6\end{bmatrix}$ Let $A=\begin{bmatrix}u&v\\x&y\end{bmatrix}$

$\therefore \begin{bmatrix}u&v\\x&y\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}=\begin{bmatrix}-7&-8&-9\\2&4&6\end{bmatrix}$ $\Rightarrow \begin{bmatrix}u+4v&2u+5v&3u+6v\\x+4y&2x+5y&3x+6y\end{bmatrix}=\begin{bmatrix}-7&-8&-9\\2&4&6\end{bmatrix}$

By equality of matrices, we have :

$u+4v=-7,\ 2u+5v=-8,\ 3u+6v=-9,\ x+4y=2, 2x+5y=4, 3x+6y=6$

On solving these equations simultaneously, we get : u = 1, v = –2, x = 2, y = 0

Hence, $A=\begin{bmatrix}1&-2\\2&0\end{bmatrix}$

Q04. Let $P=\begin{bmatrix}2&-1\\1&0\\-3&4\end{bmatrix}$ and, $Q=\begin{bmatrix}-1&-8&-10\\1&-2&-5\\9&22&15\end{bmatrix}$.

Since order of P is 3 by 2 and that of matrix Q is 3 by 3 so matrix A must be of order 2×3.

Let $A=\begin{bmatrix}u&v&w\\x&y&z\end{bmatrix}$.

Now $PA = Q \quad \therefore \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2u-x & 2v-y & 2w-z \\ u & v & w \\ -3u+4x & -3v+4y & -3w+4z \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

By equality of matrices, $2u-x=-1, 2v-y=-8, 2w-z=-10, u=1, v=-2, w=-5$, $-3u+4x=9, -3v+4y=22, -3w+4z=15$

On solving these eqs. simultaneously, we get : $u=1, v=-2, w=-5, x=3, y=4, z=0$

Therefore, $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$.

Q05. **Symmetric matrix :** A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix if $A = A^T$.

That is, if $A = [a_{ij}]$ then, $A^T = [a_{ji}] = [a_{ij}] \quad \Rightarrow A^T = A$.

Skew-symmetric matrix : A square matrix $A = [a_{ij}]$ is said to be a skew-symmetric matrix if $A^T = -A$ i.e., if $A = [a_{ij}]$ then, $A^T = [a_{ji}] = -[a_{ij}] \quad \Rightarrow A^T = -A$.

Now $X = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} \qquad \Rightarrow X^T = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$

$\therefore X - X^T = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = P$ (say)

$P^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -P \Rightarrow P = -P^T \quad \therefore P = X - X^T$ is a skew-symmetric matrix.

Also, $X + X^T = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & -8 \end{bmatrix} = Q$ (say)

$\Rightarrow Q^T = \begin{bmatrix} -2 & 3 \\ 3 & -8 \end{bmatrix} = Q \qquad \therefore$ Q is symmetric matrix.

Similarly, $XX^T = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ -6 & 20 \end{bmatrix} = M$ (say)

$\Rightarrow M^T = \begin{bmatrix} 2 & -6 \\ -6 & 20 \end{bmatrix} = M \qquad \therefore$ M is a symmetric matrix.

Similarly, $X^TX = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -9 \\ -9 & 17 \end{bmatrix} = N$ (say)

$\Rightarrow N^T = \begin{bmatrix} 5 & -9 \\ -9 & 17 \end{bmatrix} = N \qquad \therefore$ N is a symmetric matrix. Hence Proved.

Q06. Let $A = \begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix} \qquad \Rightarrow A^T = \begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}$

$\therefore A + A^T = \begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 10 \end{bmatrix}$ & $A - A^T = \begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$

Let $P=\frac{1}{2}\left(A+A^T\right)=\frac{1}{2}\begin{bmatrix}4 & 3\\ 3 & 10\end{bmatrix}=\begin{bmatrix}2 & 3/2\\ 3/2 & 5\end{bmatrix}$

$\therefore P^T=\begin{bmatrix}2 & 3/2\\ 3/2 & 5\end{bmatrix}=P \quad \Rightarrow P$ is a symmetric matrix

Also $Q=\frac{1}{2}\left(A-A^T\right)=\frac{1}{2}\begin{bmatrix}0 & -5\\ 5 & 0\end{bmatrix}=\begin{bmatrix}0 & -5/2\\ 5/2 & 0\end{bmatrix}$

$\therefore Q^T=\begin{bmatrix}0 & 5/2\\ -5/2 & 0\end{bmatrix}=-\begin{bmatrix}0 & -5/2\\ 5/2 & 0\end{bmatrix}=-Q \quad \Rightarrow Q$ is a skew-symmetric matrix.

Now, $P+Q=\begin{bmatrix}2 & 3/2\\ 3/2 & 5\end{bmatrix}+\begin{bmatrix}0 & -5/2\\ 5/2 & 0\end{bmatrix}=\begin{bmatrix}2 & -1\\ 4 & 5\end{bmatrix}=A$

Hence A has been expressed as the sum of a symmetric and skew-symmetric matrix.

Q07. We have $A=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}$

Now let $P=\frac{1}{2}\left(A+A^T\right)=\frac{1}{2}\left\{\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}+\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}\right\} \qquad \Rightarrow P=\frac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$

Also, $Q=\frac{1}{2}\left(A-A^T\right)=\frac{1}{2}\left\{\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}-\begin{bmatrix}0 & -a & -b\\ a & 0 & -c\\ b & c & 0\end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix}0 & 2a & 2b\\ -2a & 0 & 2c\\ -2b & -2c & 0\end{bmatrix}$

$\Rightarrow Q=\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}$.

Q08. Given $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}, B=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}$ Let $P=AB=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}.\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}=\begin{bmatrix}1 & 5\\ 3 & 7\end{bmatrix}$

And, let $Q=BA=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}=\begin{bmatrix}1 & 3\\ 5 & 7\end{bmatrix}$

Now let $R=P-Q=AB-BA \qquad \Rightarrow R=\begin{bmatrix}1 & 5\\ 3 & 7\end{bmatrix}-\begin{bmatrix}1 & 3\\ 5 & 7\end{bmatrix}=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$

$\Rightarrow R^T=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}=-\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}=-R \qquad \therefore R$ is a skew-symmetric matrix.

Q09. **(a)** Let $A=\begin{bmatrix}3 & 2 & 3\\ 4 & 5 & 3\\ 2 & 4 & 5\end{bmatrix} \qquad \Rightarrow A^T=\begin{bmatrix}3 & 4 & 2\\ 2 & 5 & 4\\ 3 & 3 & 5\end{bmatrix}$

Let $P=\frac{1}{2}\left(A+A^T\right)=\frac{1}{2}\left\{\begin{bmatrix}3&2&3\\4&5&3\\2&4&5\end{bmatrix}+\begin{bmatrix}3&4&2\\2&5&4\\3&3&5\end{bmatrix}\right\}$ $\Rightarrow P=\begin{bmatrix}3&3&5/2\\3&5&1/2\\5/2&7/2&5\end{bmatrix}$

$\therefore P^T=\begin{bmatrix}3&3&5/2\\3&5&7/2\\5/2&7/2&5\end{bmatrix}=P$ $\therefore$ P is a symmetric matrix.

Also $Q=\frac{1}{2}\left(A-A^T\right)=\frac{1}{2}\left\{\begin{bmatrix}3&2&3\\4&5&3\\2&4&5\end{bmatrix}-\begin{bmatrix}3&4&2\\2&5&4\\3&3&5\end{bmatrix}\right\}$ $\Rightarrow Q=\begin{bmatrix}0&-1&1/2\\1&0&-1/2\\-1/2&1/2&0\end{bmatrix}$

$\therefore Q^T=\begin{bmatrix}0&1&-1/2\\-1&0&1/2\\1/2&-1/2&0\end{bmatrix}=-\begin{bmatrix}0&-1&1/2\\1&0&-1/2\\-1/2&1/2&0\end{bmatrix}=-Q$ $\therefore$ Q is a skew-symmetric matrix.

Now $P+Q=\begin{bmatrix}3&3&5/2\\3&5&7/2\\5/2&7/2&5\end{bmatrix}+\begin{bmatrix}0&-1&1/2\\1&0&-1/2\\-1/2&1/2&0\end{bmatrix}=\begin{bmatrix}3&2&3\\4&5&3\\2&4&5\end{bmatrix}=A$

Hence A has been expressed as the sum of a symmetric and skew-symmetric matrix.

(b) Do yourself. Ans. : $\begin{bmatrix}-3&6&0\\4&-5&8\\0&-7&-1\end{bmatrix}=\begin{bmatrix}-3&5&0\\5&-5&1/2\\0&1/2&-1\end{bmatrix}+\begin{bmatrix}0&1&0\\-1&0&15/2\\0&-15/2&0\end{bmatrix}$

(c) Let $A=\begin{bmatrix}2&4&-6\\7&3&5\\1&-2&4\end{bmatrix}$ $\Rightarrow A^T=\begin{bmatrix}2&7&1\\4&3&-2\\-6&5&4\end{bmatrix}$

Let $P=\frac{1}{2}(A+A^T)=\frac{1}{2}\begin{bmatrix}2&4&-6\\7&3&5\\1&-2&4\end{bmatrix}+\frac{1}{2}\begin{bmatrix}2&7&1\\4&3&-2\\-6&5&4\end{bmatrix}=\begin{bmatrix}2&11/2&-5/2\\11/2&3&3/2\\-5/2&3/2&4\end{bmatrix}$

$\Rightarrow P^T=\begin{bmatrix}2&11/2&-5/2\\11/2&3&3/2\\-5/2&3/2&4\end{bmatrix}$ $\because P=P^T$ $\therefore$ P is symmetric matrix.

Also let $Q=\frac{1}{2}(A-A^T)=\frac{1}{2}\begin{bmatrix}2&4&-6\\7&3&5\\1&-2&4\end{bmatrix}-\frac{1}{2}\begin{bmatrix}2&7&1\\4&3&-2\\-6&5&4\end{bmatrix}=\begin{bmatrix}0&-3/2&-7/2\\3/2&0&7/2\\7/2&-7/2&0\end{bmatrix}$

$\Rightarrow Q^T=\begin{bmatrix}0&3/2&7/2\\-3/2&0&-7/2\\-7/2&7/2&0\end{bmatrix}=-Q$ $\because Q=-Q^T$ $\therefore$ Q is skew-symmetric matrix.

Hence $P+Q=\begin{bmatrix}2 & 11/2 & -5/2\\ 11/2 & 3 & 3/2\\ -5/2 & 3/2 & 4\end{bmatrix}+\begin{bmatrix}0 & -3/2 & -7/2\\ 3/2 & 0 & 7/2\\ 7/2 & -7/2 & 0\end{bmatrix}=A$.

Q10. $A=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$ and, $f(x)=x^2-5x-6$

Now, $A^2=A.A=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}=\begin{bmatrix}14 & 10\\ -5 & -1\end{bmatrix}$... (i)

$\therefore f(A)=A^2-5(A)-6I=\begin{bmatrix}14 & 10\\ -5 & -1\end{bmatrix}-5\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}-6\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ [By (i)

$\Rightarrow \quad =\begin{bmatrix}14 & 10\\ -5 & -1\end{bmatrix}-\begin{bmatrix}20 & 10\\ -5 & 5\end{bmatrix}-\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}=\begin{bmatrix}-12 & 0\\ 0 & -12\end{bmatrix}=-12I$.

Q11. We have $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$ and $f(x)=x^2-5x+6\ldots$(i)

Now $A^2=A.A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}=\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}$...(ii)

$\therefore f(A)=A^2-5A+6I=\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}-5\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}+6\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ [By (ii)

$\Rightarrow \quad =\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}-\begin{bmatrix}10 & 0 & 5\\ 10 & 5 & 15\\ 5 & -5 & 0\end{bmatrix}+\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{bmatrix}$ Therefore, $f(A)=\begin{bmatrix}1 & -1 & -3\\ -1 & -1 & -10\\ -5 & 4 & 4\end{bmatrix}$

Q12. If A is a root of the given cubic equation then it must satisfy $x^3-6x^2+7x+2=0$.
i.e., $A^3-6A^2+7A+2I=O\ldots$(i)

Now, $A^3=A.A.A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$ [Given $A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$

$\Rightarrow A^2.A=\begin{bmatrix}5 & 0 & 8\\ 2 & 4 & 5\\ 8 & 0 & 13\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$ $\Rightarrow A^3=\begin{bmatrix}21 & 0 & 34\\ 12 & 8 & 23\\ 34 & 0 & 55\end{bmatrix}$

$\therefore A^3-6A^2+7A+2I=\begin{bmatrix}21 & 0 & 34\\ 12 & 8 & 23\\ 34 & 0 & 55\end{bmatrix}-6\begin{bmatrix}5 & 0 & 8\\ 2 & 4 & 5\\ 8 & 0 & 13\end{bmatrix}+7\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}+\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$

$\Rightarrow \quad =\begin{bmatrix}23 & 0 & 34\\ 12 & 10 & 23\\ 34 & 0 & 57\end{bmatrix}-\begin{bmatrix}30 & 0 & 48\\ 12 & 24 & 30\\ 48 & 0 & 78\end{bmatrix}+\begin{bmatrix}7 & 0 & 14\\ 0 & 14 & 7\\ 14 & 0 & 21\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$

$\because$ A satisfies the given equation, so it is a root of given cubic equation.

Q13. (a) $A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}$ $\Rightarrow A^2=A.A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}=\begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix}$

$\therefore A^2=kA-2I \Rightarrow \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix}=k\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$ $\Rightarrow \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix}=\begin{bmatrix}3k-2 & -2k\\ 4k & -4\end{bmatrix}$

By equality of matrices, $4k=4$ $\Rightarrow k=1$.

(b) $A=\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$ $\because A^{-1}=kA$ $\Rightarrow A^{-1}A=kAA$ $\therefore I=kA^2$

So, $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=k\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$ $\Rightarrow \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=k\begin{bmatrix}19 & 0\\ 0 & 19\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}19k & 0\\ 0 & 19k\end{bmatrix}$ By equality of matrices, $1=19k$ $\Rightarrow k=\frac{1}{19}$

(c) $A=\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$ So, $\text{Adj } A=\begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix}$, $|A|=-4-15=-19$

As $A^{-1}=\frac{1}{|A|}\times(\text{adj}A)$ $\therefore A^{-1}=\frac{1}{-19}\begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix}$

So, $A^{-1}=\frac{1}{19}\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$ Hence, $A^{-1}=\frac{1}{19}A$

Note that, here we can use the method used in Q13 (b) as well. We've elaborated **two different approaches** to solve similar Q13 (b) and (c).

Q14. **(a)** Given $f(x)=x^2-4x+7$ To show : $f(A)=A^2-4A+7I=O$

Now $A^2=A.A=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}=\begin{bmatrix}1 & 12\\ -4 & 1\end{bmatrix}$

Consider $f(A)=A^2-4A+7I=\begin{bmatrix}1 & 12\\ -4 & 1\end{bmatrix}-4\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \quad =\begin{bmatrix}8 & 12\\ -4 & 8\end{bmatrix}-\begin{bmatrix}8 & 12\\ -4 & 8\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$ Hence, $f(A)=A^2-4A+7I=O$.

Now $A^2=4A-7I \ldots$(i) $\Rightarrow A^2A=4AA-7IA$ [Post-multiplying by A both the sides

$\Rightarrow A^3=4A^2-7A$ $\Rightarrow A^3=4(4A-7I)-7A=9A-28I$ [By (i)]

$\Rightarrow A^3A=9AA-28IA$ $\Rightarrow A^4=9(4A-7I)-28A=8A-63I$

$\Rightarrow A^4A=8AA-63IA$ $\therefore A^5=8(4A-7I)-63A=-31A-56I$

$\Rightarrow A^5=-31\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}-56\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ Hence, $A^5=\begin{bmatrix}-118 & -93\\ 31 & -118\end{bmatrix}$.

(b) Here $A=\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}$ $\Rightarrow A^2=A.A=\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}1 & 12\\ -4 & 1\end{pmatrix}$

$\therefore A^2-4A+7I=\begin{pmatrix}1 & 12\\ -4 & 1\end{pmatrix}-4\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}+7\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=O$

Now $A^2-4A+7I=O \Rightarrow A^2=4A-7I \Rightarrow A^3=A.A^2=4A^2-7A=4(4A-7I)-7A=9A-28I$

$\therefore A^3=9A-28I=\begin{pmatrix}18 & 27\\ -9 & 18\end{pmatrix}-\begin{pmatrix}28 & 0\\ 0 & 28\end{pmatrix}=\begin{pmatrix}-10 & 27\\ -9 & -10\end{pmatrix}$

Q15. Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and, $A^2 - 2B + 7I = O$

Now $A^2 = A.A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5I$

$\therefore A^2 - 2B + 7I = O \quad \Rightarrow 5I - 2B + 7I = O \quad \therefore B = 6I = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

Q16. Do yourself. Proceed as in Q12.

Q17. **(a)** LHS: Let $(A-5I)(A-2I) = A.A - 2AI - 5AI + 10I$

$\Rightarrow \quad = A^2 - 7A + 10I \ldots$(i)

Now, $A^2 = A.A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 11 & 14 & 0 \\ 7 & 18 & 0 \\ 0 & 0 & 25 \end{bmatrix}$ [Given $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

By (i), $A^2 - 7A + 10I = \begin{bmatrix} 11 & 14 & 0 \\ 7 & 18 & 0 \\ 0 & 0 & 25 \end{bmatrix} - 7\begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \quad = \begin{bmatrix} 21 & 14 & 0 \\ 7 & 28 & 0 \\ 0 & 0 & 35 \end{bmatrix} - \begin{bmatrix} 21 & 14 & 0 \\ 7 & 28 & 0 \\ 0 & 0 & 35 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = RHS$

Now $A^2 - 7A + 10I = O \quad \Rightarrow AAA^{-1} - 7AA^{-1} + 10IA^{-1} = OA^{-1} \quad \Rightarrow AI - 7I + 10A^{-1} = O$

$\Rightarrow 10A^{-1} = 7I - A = 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 2/5 & -1/5 & 0 \\ -1/10 & 3/10 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$

(b) We have $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

$\Rightarrow A^2 = A.A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$

LHS : $A^2 - 5A + 4I = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 5\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \quad = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - \begin{pmatrix} 10 & -5 & 5 \\ -5 & 10 & -5 \\ 5 & -5 & 10 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O = RHS$

Now $A^2 - 5A + 4I = O \quad \Rightarrow A^2 - 5A + 4I = O$ [Pre-multiplying both the sides by A^{-1}

$\Rightarrow A^{-1}AA - 5A^{-1}A + 4A^{-1}I = A^{-1}O \quad \Rightarrow A^{-1} = \dfrac{5I - A}{4}$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[5\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} - \begin{pmatrix}2 & -1 & 1\\-1 & 2 & -1\\1 & -1 & 2\end{pmatrix}\right] \qquad \therefore A^{-1} = \frac{1}{4}\begin{pmatrix}3 & 1 & -1\\1 & 3 & 1\\-1 & 1 & 3\end{pmatrix}.$$

(c) We have $A = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} \Rightarrow A^2 = A.A = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} = \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix}$

Now LHS : $A^2 - 4A - 5I = \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix} - 4\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} - 5\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = O = RHS.$

Also $A^2 - 4A - 5I = O \qquad \Rightarrow A^{-1}AA - 4A^{-1}A - 5A^{-1}I = A^{-1}O$

$\Rightarrow IA - 4I - 5A^{-1} = O$

$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I) \qquad \therefore A^{-1} = \frac{1}{5}\left\{\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} - 4\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix}-3 & 2 & 2\\2 & -3 & 2\\2 & 2 & -3\end{bmatrix}.$

Q18. We have $A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix}$

Now, $A^2 = A.A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix} = \begin{bmatrix}4 & 2 & 1\\-3 & 8 & -14\\7 & -3 & 14\end{bmatrix}$

Also, $A^3 = A^2.A = \begin{bmatrix}4 & 2 & 1\\-3 & 8 & -14\\7 & -3 & 14\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix} = \begin{bmatrix}8 & 7 & 1\\-23 & 27 & -69\\32 & -13 & 58\end{bmatrix}$

Consider LHS : $A^3 - 6A^2 + 5A + 11I$

$\Rightarrow \quad = \begin{bmatrix}8 & 7 & 1\\-23 & 27 & -69\\32 & -13 & 58\end{bmatrix} - \begin{bmatrix}24 & 12 & 6\\-18 & 48 & -84\\42 & -18 & 84\end{bmatrix} + \begin{bmatrix}5 & 5 & 5\\5 & 10 & -15\\10 & -5 & 15\end{bmatrix} + \begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}$

$\Rightarrow \quad = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = O = RHS.$

Now $A^3 - 6A^2 + 5A + 11I = O$ [Pre-multiplying by A^{-1} on both sides

$\Rightarrow A^{-1}AAA - 6A^{-1}AA + 5A^{-1}A + 11A^{-1}I = A^{-1}O \qquad \Rightarrow A^2 - 6A + 5I + 11A^{-1} = O$

$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I \qquad \therefore 11A^{-1} = -\begin{bmatrix}4 & 2 & 1\\-3 & 8 & -14\\7 & -3 & 14\end{bmatrix} + \begin{bmatrix}6 & 6 & 6\\6 & 12 & -18\\12 & -6 & 18\end{bmatrix} - \begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix}$

$\Rightarrow 11A^{-1} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$ $\quad \therefore A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

Q19. **(a)** Let $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$. So, $A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$

If A satisfies the equation $x^2 - 3x - 7 = 0$, then $A^2 - 3A - 7I = O$.

Now LHS: $A^2 - 3A - 7I = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O = RHS.$

Now, $A^2 - 3A - 7I = O$ [Pre-multiplying by A^{-1} both sides

$\Rightarrow A^{-1}A.A - 3A^{-1}A - 7A^{-1}I = A^{-1}O$ $\quad \Rightarrow IA - 3I - 7A^{-1} = O$ $\quad \Rightarrow 7A^{-1} = A - 3I$

$\therefore 7A^{-1} = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad$ Hence, $A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

(b) We have $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\quad \Rightarrow A^2 = A.A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$…(i)

Also $4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$...(ii)

By (i) and (ii), we get : $A^2 = 4A - 3I$.

Pre-multiplying both sides by A^{-1} we get : $A^{-1}AA = 4A^{-1}A - 3A^{-1}I$ $\quad \Rightarrow IA = 4I - 3A^{-1}$

$\Rightarrow 3A^{-1} = 4I - A = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \therefore A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Q20. **(a)** Given $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$. So, $A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$

Since A satisfies $A^2 + xI - yA = O$ so, $\begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = O$

$\Rightarrow \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$ $\quad \Rightarrow \begin{bmatrix} 3y - x & y \\ 7y & 5y - x \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$

By equality of matrices, $y = 8$ and $3y - x = 16$ $\Rightarrow x = 24 - 16 = 8$ $\quad \therefore x = 8$ and, $y = 8$

$\therefore A^2 + 8I - 8A = O$ [Pre-multiplying by A^{-1} both sides

$\Rightarrow A^{-1}AA + 8A^{-1}I - 8A^{-1}A = A^{-1}O$ $\quad \Rightarrow IA + 8A^{-1} - 8I = O$ $\quad \Rightarrow 8A^{-1} = 8I - A$

$\Rightarrow 8A^{-1} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$ $\quad \therefore A^{-1} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$

(b) Do yourself. Ans. $\lambda = 4$ and $\mu = 3$.

Q21. Given $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad \Rightarrow A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

$\therefore A^2 + aA + bI = O$ $\quad \Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$

$$\Rightarrow \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} -11 & -8 \\ -4 & -3 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} 3a+b & 2a \\ a & a+b \end{bmatrix} = \begin{bmatrix} -11 & -8 \\ -4 & -3 \end{bmatrix}$$

By equality of matrices, $a=-4$ & $a+b=-3$ $\quad \therefore a=-4,\ b=1$.

$\therefore A^2-4A+1.I=O$ [Pre-multiplying by A^{-1} both sides

$\Rightarrow A^{-1}A.A-4A^{-1}A+A^{-1}I=A^{-1}O \quad \Rightarrow I.A-4I+A^{-1}=O \quad \Rightarrow A^{-1}=4I-A$

$$\therefore A^{-1}=\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}-\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Q22. Do yourself.

Q23. (a) Given $A=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Let $P(n): A^n=\begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

For n = 1, $P(1)=A^1=\begin{bmatrix} \cos(1)\theta & \sin(1)\theta \\ -\sin(1)\theta & \cos(1)\theta \end{bmatrix} \quad \Rightarrow A=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \therefore P(1)$ is true.

Let us assume that P(k) is true $\forall k\in N$ i.e, $P(k): A^K=\begin{bmatrix} \cos k\theta & \sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$ ...(i)

We have to prove that $P(k+1)$ is also true whenever $P(k)$ is true.

i.e., $P(k+1): A^{k+1}=\begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \text{cons}(k+1)\theta \end{bmatrix}$

LHS : $A^{k+1}=A^k.A=\begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ [By (i)

$$=\begin{bmatrix} \cos k\theta\cos\theta-\sin k\theta\sin\theta & \cos k\sin\theta+\sin k\theta\cos\theta \\ -(\sin k\theta\cos\theta+\sin\theta\cos k\theta) & \cos\theta\cos k\theta-\sin k\theta\sin\theta \end{bmatrix}$$

$$=\begin{bmatrix} \cos(k\theta+\theta) & \sin(k\theta+\theta) \\ -\sin(k\theta+\theta) & \cos(k\theta+\theta) \end{bmatrix}$$

$\Rightarrow A^{k+1}=\begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \text{cons}(k+1)\theta \end{bmatrix}=\text{RHS} \quad \therefore P(k+1)$ is also true.

Hence we can say that by using Principle of Mathematical Induction this statement, $P(n)$ will always be true for all $n\in N$.

(b) Given $A=\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$

Let $P(n): A^n=\begin{bmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{bmatrix}$

For $n=1$, $P(1): A^1=\begin{bmatrix} \cos(1)\theta & i\sin(1)\theta \\ i\sin(1)\theta & \cos(1)\theta \end{bmatrix} \quad \Rightarrow A=\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix} \quad \therefore P(1)$ is true.

Let us assume that $P(K)$ is true $\forall K\in N$ i.e., $P(K): A^K=\begin{bmatrix} \cos K\theta & i\sin K\theta \\ i\sin K\theta & \cos K\theta \end{bmatrix}$ …(i)

Now, we have to prove that $P(K+1)$ is also true whenever $P(K)$ is true.

That is, $P(K+1): A^{K+1} = \begin{bmatrix} \cos(K+1)\theta & i\sin(K+1)\theta \\ i\sin(K+1)\theta & \cos(K+1)\theta \end{bmatrix}$

$LHS: A^{K+1} = A^K A = \begin{bmatrix} \cos K\theta & i\sin K\theta \\ i\sin K\theta & \cos K\theta \end{bmatrix}\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$ [By (i)

$= \begin{bmatrix} \cos K\theta\cos\theta + i^2\sin K\theta\sin\theta & i(\cos K\theta\sin\theta + \sin K\theta\cos\theta) \\ i(\sin K\theta\cos\theta + \sin\theta\cos K\theta) & \cos K\theta\cos\theta + i^2\sin K\theta\sin\theta \end{bmatrix}$

$= \begin{bmatrix} \cos K\theta\cos\theta - \sin K\theta\sin\theta & i\sin(K\theta+\theta) \\ i\sin(K\theta+\theta) & \cos K\theta cs\theta - \sin K\theta\sin\theta \end{bmatrix}$ $[\because i^2 = -1$

$= \begin{bmatrix} \cos(K\theta+\theta) & i\sin(K\theta+\theta) \\ i\sin(K\theta+\theta) & \cos(K\theta+\theta) \end{bmatrix}$

$\Rightarrow A^{K+1} = \begin{bmatrix} \cos(K+1)\theta & i\sin(K+1)\theta \\ i\sin(K+1)\theta & \cos(K+1)\theta \end{bmatrix} = RHS$ $\therefore P(K+1)$ is also true.

Hence by using principle of mathematical induction, we can say that this statement P(n) is always true for all natural numbers n.

(c) Given $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Let $P(n): (aI + bA)^n = a^n I + na^{n-1}bA$

For n = 1, $P(1): (aI+bA)^1 = a^1 I + (1)a^{1-1}bA$ $\Rightarrow (aI+bA) = aI + bA$ $\therefore P(1)$ is true.

Let us assume that $P(K)$ is true $\forall K \in N$, i.e., $P(K): (aI+bA)^K = a^K I + Ka^{K-1}bA$ …(i)

Now, we have to prove that $P(K+1)$ is also true whenever $P(K)$ is true,

i.e., $P(K+1) = (aI+bA)^{K+1} = a^{K+1}I + (K+1)a^K bA$

LHS : $(aI+bA)^{K+1} = (aI+bA)^K (aI+bA)$

$\Rightarrow \quad = (a^K I + Ka^{K-1}bA)(aI+bA)$ [By (i)

$\Rightarrow \quad = a^{K+1}I + a^K bAI + Ka^K bA + Ka^{K-1}b^2A^2$

$\Rightarrow \quad = a^{K+1}I + a^K bA(K+1) + O$ $\left[\because A^2 = A.A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = O\right.$

$\Rightarrow \quad = a^{K+1}I + (K+1)a^K bA = RHS$ $\therefore P(K+1)$ is also true.

Hence we can say that by using Principle of Mathematical Induction this statement, $P(n)$ will always be true for all $n \in N$.

(d) Given $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

Let $P(n): A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}, n \in Z^+$

For $n = 1$, $P(1): A^1 = \begin{bmatrix} 1+2(1) & -4(1) \\ 1 & 1-2(1) \end{bmatrix}$ $\Rightarrow A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ $\therefore P(1)$ is true.

Let us assume that $P(K)$ is true for all $K \in Z^+$.

i.e., $P(K): A^K = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ ...(i)

We've to prove that $P(K+1)$ is also true whenever $P(K)$ is true,

i.e., $P(K+1): A^{K+1} = \begin{bmatrix} 1+2(K+1) & -4(K+1) \\ K+1 & 1-2(K+1) \end{bmatrix}$

LHS: $A^{K+1} = A^K.A = \begin{bmatrix} 1+2K & -4K \\ K & 1-2K \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ [By (i)

$\Rightarrow \quad = \begin{bmatrix} 3+6K-4K & -4-8K+4K \\ 3K+1-2K & -4K-1+2K \end{bmatrix} \quad \Rightarrow \quad = \begin{bmatrix} 1+2(K+1) & -4(K+1) \\ K+1 & 1-2(K+1) \end{bmatrix} = \text{RHS}$

$\therefore P(K+1)$ is also true.

Hence we can say that by using Principle of Mathematical Induction this statement, $P(n)$ will always be true for all $n \in Z^+$.

(e) Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Let $P(n): A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$

For $n=1, P(1): A^1 = \begin{bmatrix} 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \end{bmatrix} = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\therefore P(1)$ is true.

Let us assume that $P(K)$ is true $\forall K \in N$, i.e. $P(K): A^K = \begin{bmatrix} 3^{K-1} & 3^{K-1} & 3^{K-1} \\ 3^{K-1} & 3^{K-1} & 3^{K-1} \\ 3^{K-1} & 3^{K-1} & 3^{K-1} \end{bmatrix}$ ...(i)

Now we have to prove that $P(K+1)$ is also true whenever $P(K)$ is true

i.e., $P(K+1): A^{K+1} = \begin{bmatrix} 3^K & 3^K & 3^K \\ 3^K & 3^K & 3^K \\ 3^K & 3^K & 3^K \end{bmatrix}$

LHS: $A^{K+1} = A^K.A = \begin{bmatrix} 3^{K-1} & 3^{K-1} & 3^{K-1} \\ 3^{K-1} & 3^{K-1} & 3^{K-1} \\ 3^{K-1} & 3^{K-1} & 3^{K-1} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ [By (i)

$\Rightarrow \quad = \begin{bmatrix} 3^{K-1}+3^{K-1}+3^{K-1} & 3^{K-1}+3^{K-1}+3^{K-1} & 3^{K-1}+3^{K-1}+3^{K-1} \\ 3^{K-1}+3^{K-1}+3^{K-1} & 3^{K-1}+3^{K-1}+3^{K-1} & 3^{K-1}+3^{K-1}+3^{K-1} \\ 3^{K-1}+3^{K-1}+3^{K-1} & 3^{K-1}+3^{K-1}+3^{K-1} & 3^{K-1}+3^{K-1}+3^{K-1} \end{bmatrix}$

$\Rightarrow \quad = \begin{bmatrix} 3\times3^{K-1} & 3\times3^{K-1} & 3\times3^{K-1} \\ 3\times3^{K-1} & 3\times3^{K-1} & 3\times3^{K-1} \\ 3\times3^{K-1} & 3\times3^{K-1} & 3\times3^{K-1} \end{bmatrix} \quad \Rightarrow A^{k+1} = \begin{bmatrix} 3^K & 3^K & 3^K \\ 3^K & 3^K & 3^K \\ 3^K & 3^K & 3^K \end{bmatrix} = \text{RHS}$

$\therefore P(K+1)$ is also true.

Hence we can say that by using Principle of Mathematical Induction this statement, $P(n)$ will always be true for all natural numbers n.

(f) Given $A=\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

Let $P(n): A^n=\begin{bmatrix} a^n & \dfrac{b(a^n-1)}{a-1} \\ 0 & 1 \end{bmatrix}$

For n = 1, $P(1): A^1=\begin{bmatrix} a^1 & \dfrac{b(a^1-1)}{a-1} \\ 0 & 1 \end{bmatrix}$ $\Rightarrow A=\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ $\therefore P(1)$ is true.

Now, let us assume that $P(K)$ is true $\forall\ K \in N$, i.e., $P(K): A^K=\begin{bmatrix} a^K & \dfrac{b(a^K-1)}{a-1} \\ 0 & 1 \end{bmatrix}$ …(i)

Now, we have to prove $P(K+1)$ is also true whenever $P(K)$ is true.

i.e., $P(K+1): A^{K+1}=\begin{bmatrix} a^{K+1} & \dfrac{b(a^{K+1}-1)}{a-1} \\ 0 & 1 \end{bmatrix}$

LHS: $A^{K+1}=A^K.A=\begin{bmatrix} a^K & \dfrac{b(a^K-1)}{a-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad =\begin{bmatrix} a^{K+1}+0 & a^K.b+\dfrac{b(a^K-1)}{a-1} \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad =\begin{bmatrix} a^{K+1}+0 & a^K.b+\dfrac{b(a^K-1)}{a-1} \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow A^{K+1}=\begin{bmatrix} a^{K+1} & \dfrac{b(a^{K+1}-1)}{a-1} \\ 0 & 1 \end{bmatrix}=\text{RHS}$ $\therefore P(K+1)$ is also true.

Hence by using principle of Mathematical induction we can say that this statement $P(n)$ is always true for all $n \geq 0$, $n \in N$.

(g) Given $A=\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$

Let $P(n): A^n=\begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}$

For n = 1, $P(1): A^1=\begin{bmatrix} a^1 & 1.a^{1-1} \\ 0 & a^1 \end{bmatrix}$ $\Rightarrow A=\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ $\therefore P(1)$ is true.

Let us assume that $P(K)$ is true for all $K \in Z^+$, i.e, $P(K): A^K=\begin{bmatrix} a^K & Ka^{K-1} \\ 0 & a^K \end{bmatrix}$...(i)

Now, we have to prove that $P(K+1)$ is also true whenever $P(K)$ is true

i.e., $P(K+1): A^{K+1} = \begin{bmatrix} a^{K+1} & (K+1)a^K \\ 0 & a^{K+1} \end{bmatrix}$

LHS: $A^{K+1} = A^K.A = \begin{bmatrix} a^K & Ka^{K-1} \\ 0 & a^K \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ [By (i)

$\Rightarrow \quad = \begin{bmatrix} a^{K+1} & (K+1)a^K \\ 0 & a^{K+1} \end{bmatrix} = \text{RHS}$ $\therefore P(K+1)$ is also true.

Hence by using principle of mathematical induction we can say that the statement P (n) is always true for all $n \in Z^+$.

(h) Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Let $P(n): A^n = \begin{bmatrix} 1 & n & \frac{n[n+1]}{2} \\ 0 & 1 & n \\ 0 & 1 & 1 \end{bmatrix}$

For $n = 1$, $P(1): A^1 = \begin{bmatrix} 1 & 1 & \frac{1[1+1]}{2} \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\therefore P(1)$ is true.

Now, let us assume that $P(K)$ is true $\forall K \in N$, i.e., $P(K) = A^K = \begin{bmatrix} 1 & K & \frac{K[K+1]}{2} \\ 0 & 1 & K \\ 0 & 0 & 1 \end{bmatrix}$ …(i)

Now, we have to show that $P(K+1)$ is also true whenever $P(K)$ is true

i.e., $P(K+1): A^{K+1} = \begin{bmatrix} 1 & K+1 & \frac{(K+1)(K+2)}{2} \\ 0 & 1 & K+1 \\ 0 & 0 & 1 \end{bmatrix}$

LHS: $A^{K+1} = A^K.A = \begin{bmatrix} 1 & K & \frac{K(K+1)}{2} \\ 0 & 1 & K \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ [By (i)

$\Rightarrow \quad = \begin{bmatrix} 1 & K+1 & 1+K+\frac{K(K+1)}{2} \\ 0 & 1 & K+1 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad = \begin{bmatrix} 1 & K+1 & \frac{(K+1)(K+2)}{2} \\ 0 & 1 & K+1 \\ 0 & 0 & 1 \end{bmatrix} = \text{RHS}$

$\therefore P(K+1)$ is also true.

Hence by using principle of mathematical induction we can say that this result, $P(n)$ will always be true $\forall\ n \in N$.

(i) Given $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Let $P(n): A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

For $n=1$, $P(1): A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \therefore P(1)$ is true.

Now, let us assume that $P(K)$ is also true $\forall\ K \in Z^+$, i.e, $P(K): A^K = \begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix}$ …(i)

Now, we have to prove that $P(K+1)$ is also true whenever $P(K)$ is true

i.e., $P(K+1): A^{K+1} = \begin{bmatrix} 1 & K+1 \\ 0 & 1 \end{bmatrix}$

LHS: $A^{K+1} = A^K.A = \begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ [By (i)

$\Rightarrow \quad = \begin{bmatrix} 1 & K+1 \\ 0 & 1 \end{bmatrix} = \text{RHS} \quad \therefore P(k+1)$ is also true.

Hence by using principle of mathematical induction, we can say that the statement $P(n)$ is always true $\forall\ n \in Z^+$.

Q24. (a) $A = \text{diag}(a \quad b \quad c) \quad \Rightarrow A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Let $P(n): A^n = \text{diag}(a^n \quad b^n \quad c^n) = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$

For $n=1$, $P(1): A^1 = \begin{bmatrix} a^1 & 0 & 0 \\ 0 & b^1 & 0 \\ 0 & 0 & c^1 \end{bmatrix} \quad \Rightarrow A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \quad \therefore P(1)$ is true.

Now, let us assume that $P(k)$ is true $\forall\ k \in Z^+$, i.e., $P(k): A^k = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}$ … (i)

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true i.e., $P(k+1): A^{k+1} = \begin{bmatrix} a^{k+1} & 0 & 0 \\ 0 & b^{k+1} & 0 \\ 0 & 0 & c^{k+1} \end{bmatrix}$

LHS: $A^{k+1} = A^k.A = \begin{bmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ [By (i)

$\Rightarrow \quad = \begin{bmatrix} a^{k+1} & 0 & 0 \\ 0 & b^{k+1} & 0 \\ 0 & 0 & c^{k+1} \end{bmatrix} = \text{RHS}$ $\therefore P(k+1)$ is also true

Hence by using principle of mathematical induction we can say that the statement P(n) is always true for all positive integers n.

(b) Let $P(n): AB^n = B^nA,\ n \in N$.

For $n=1,\ P(1): AB^1 = B^1A \ \Rightarrow P(1): AB = BA$ [$\because$ Given AB = BA

$\therefore P(1)$ is true.

Now, let us assume that $P(k)$ is true $\forall\ k \in N$, i.e., $P(k): AB^k = B^kA$ … (i)

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true

i.e., $P(k+1): AB^{k+1} = B^{k+1}A$

LHS: $AB^{k+1} = AB^k.B = (AB^k).B = (B^kA).B$ [By (i)

$\Rightarrow \quad = B^k(AB) = B^k(BA) = B^{k+1}A = \text{RHS}$ $\therefore P(k+1)$ is also true

Hence by using principle of mathematical induction we can say that the statement P(n) is always true for all natural numbers n.

Also, let $P(n): (AB)^n = A^nB^n,\ n \in N$.

For $n=1,\ P(1): (AB)^1 = A^1B^1 \ \Rightarrow P(1): AB = BA$ $\therefore P(1)$ is true.

Now, let us assume that $P(k)$ is true $\forall\ k \in N$, i.e., $P(k): (AB)^k = A^kB^k$ … (ii)

Now, we have to prove that $P(k+1)$ is also true whenever $P(k)$ is true

i.e., $P(k+1): (AB)^{k+1} = A^{k+1}B^{k+1}$

LHS: $(AB)^{k+1} = (AB)^k.(AB) = (A^kB^k)(AB)$ [By (ii)

$\Rightarrow \quad = A^k(B^kA)B$ [By Associative law

$\Rightarrow \quad = A^k(AB^k)B$ [$\because AB^n = B^nA\ \forall\, n \in N$

$\Rightarrow \quad = (A^kA)(B^kB) = A^{k+1}B^{k+1} = \text{RHS}$ $\therefore P(k+1)$ is also true

Hence by using principle of mathematical induction we can say that the statement P(n) is always true for all natural numbers n.

Q25. Method 1 : LHS: $I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan(x/2) \\ \tan(x/2) & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\tan(x/2) \\ \tan(x/2) & 1 \end{bmatrix}$ …(i)

RHS: $(I - A)\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} = \begin{bmatrix} 1 & \tan(x/2) \\ -\tan(x/2) & 1 \end{bmatrix}\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$

$$\Rightarrow \quad = \begin{bmatrix} \cos x + \tan(x/2)\sin x & -\sin x + \tan(x/2)\cos x \\ -\tan(x/2)\cos x + \sin x & \tan(x/2)\sin x + \cos x \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} \cos x + \dfrac{\sin(x/2)}{\cos(x/2)}\sin x & -\sin x + \dfrac{\sin(x/2)}{\cos(x/2)}\cos x \\ -\dfrac{\sin(x/2)}{\cos(x/2)}\cos x + \sin x & \dfrac{\sin(x/2)}{\cos(x/2)}\sin x + \cos x \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} \dfrac{\cos x\cos(x/2) + \sin(x/2)\sin x}{\cos(x/2)} & \dfrac{\cos x\sin(x/2) - \sin x\cos(x/2)}{\cos(x/2)} \\ \dfrac{\sin x\cos(x/2) - \cos x\sin(x/2)}{\cos(x/2)} & \dfrac{\cos x\cos(x/2) + \sin(x/2)\sin x}{\cos(x/2)} \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} \dfrac{\cos(x/2)}{\cos(x/2)} & -\dfrac{\sin(x/2)}{\cos(x/2)} \\ \dfrac{\sin(x/2)}{\cos(x/2)} & \dfrac{\cos(x/2)}{\cos(x/2)} \end{bmatrix} \quad \Rightarrow \quad = \begin{bmatrix} 1 & -\tan(x/2) \\ \tan(x/2) & 1 \end{bmatrix} \qquad \ldots(ii)$$

By (i) and (ii), we observe that : LHS = RHS.

Method 2 : Use $\cos x = \dfrac{1-\tan^2\left(\dfrac{x}{2}\right)}{1+\tan^2\left(\dfrac{x}{2}\right)}$, $\sin x = \dfrac{2\tan\left(\dfrac{x}{2}\right)}{1+\tan^2\left(\dfrac{x}{2}\right)}$ and complete yourself.

Q26. We have $\phi(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore \phi(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, LHS : $\phi(x)\phi(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad = \begin{bmatrix} \cos x\cos y - \sin x\sin y & -(\cos x\sin y + \sin x\cos y) & 0 \\ \sin x\cos y + \cos x\sin y & (\cos x\cos y - \sin x\sin y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{RHS}$$

Q27. (a) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

By using elementary row transformations, we have $A = IA$

That is, $A\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}A$ By $R_2 \to R_2 - 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}A$ By $R_2 \to -\dfrac{1}{5}R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2/5 & -1/5 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 - 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix} A \qquad \because I = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

(b) Let $A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

By using elementary row operations, $A = IA$ i.e., $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

By $R_1 \to R_1 - R_2$ $\quad \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$

By $R_2 \to R_2 - 2R_1$ $\quad \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} A$

By $R_1 \to R_1 - 3R_2$ $\quad \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} A$

$\because I = A^{-1}A \qquad \therefore A^{-1} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$

(c) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

By using elementary column transformations $A = AI$

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{By } C_2 \to C_2 + C_1$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} = A\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{By } C_1 \to C_1 - \frac{2}{5}C_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = A\begin{bmatrix} 3/5 & 1 \\ -2/5 & 1 \end{bmatrix} \qquad \text{By } C_2 \to \frac{1}{5}C_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A\begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix} \qquad \because I = AA^{-1}$$

$$A^{-1} = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

(d) Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

By using elementary row operations, $A = IA$

$$\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 + 3R_2$$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A \qquad \because$ all the elements of R_1 are zero , $\therefore A^{-1}$ does not exist.

(e) Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

By using elementary row transformations, $A = IA$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_3 \to R_3 - 3R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad \text{By } R_3 \to R_3 + 5R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 - 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \qquad \text{By } R_3 \to \frac{1}{2} R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \qquad \text{By } R_2 \to R_2 - 2R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 + R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \qquad \because I = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

(f) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

By using elementary row operations, $A = I\,A$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_3 \to R_3 - 3R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_2 \to \frac{1}{2} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3/2 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_3 \to R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ -3 & -1/2 & 1 \end{bmatrix} A \qquad \text{By } R_2 \to R_2 - 3R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 - \frac{1}{2} R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} A \qquad \because I = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

(g) Let $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

By using elementary column operations, A = A I

$$\Rightarrow \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } C_1 \to \frac{1}{2} C_1$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 1 & 2 & 3 \\ 3/2 & -2 & 2 \end{bmatrix} = A \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } C_2 \to C_2 + 3C_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 1 & 5 & 3 \\ 3/2 & 5/2 & 2 \end{bmatrix} = A \begin{bmatrix} 1/2 & 3/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } C_3 \to C_3 - 3C_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 3/2 & 5/2 & -5/2 \end{bmatrix} = A \begin{bmatrix} 1/2 & 3/2 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } C_1 \to C_1 - \frac{1}{5}C_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 5/2 & -5/2 \end{bmatrix} = A \begin{bmatrix} 1/5 & 3/2 & -3/2 \\ -1/5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } C_2 \to \frac{1}{5}C_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1/2 & -5/2 \end{bmatrix} = A \begin{bmatrix} 1/5 & 3/10 & -3/2 \\ -1/5 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } C_2 \to C_2 + \frac{1}{5}C_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -5/2 \end{bmatrix} = A \begin{bmatrix} 1/5 & 0 & -3/2 \\ -1/5 & 1/5 & 0 \\ 0 & 1/5 & 1 \end{bmatrix} \qquad \text{By } C_3 \to -\frac{2}{5}C_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 0 & 1/5 & -2/5 \end{bmatrix} \qquad \text{By } C_1 \to C_1 - C_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} \qquad \because I = AA^{-1}$$

$$\therefore A^{-1} = \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

(h) Let $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

By using elementary row transformation, $A = IA$

i.e., $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_2 \to R_2 + R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_2 \to R_2 + 2R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 + R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_3 \to R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 + R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \qquad \because I = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

(i) Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$

By using elementary column transformations, A = A I

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } C_1 \to C_1 - C_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ -2 & 4 & 5 \\ 5 & -4 & -7 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \text{By } C_3 \to C_3 + C_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 3 \\ 5 & -4 & -2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \text{By } C_2 \to \frac{1}{4}C_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 5 & -1 & -2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1/4 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \text{By } C_1 \to C_1 + 2C_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 3 & -1 & -2 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1/4 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \text{By } C_3 \to C_3 - 3C_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1/4 & -3/4 \\ -1 & 0 & 0 \end{bmatrix} \qquad \text{By } C_1 \to C_1 - 3C_3,\ C_2 \to C_2 + C_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 1 \\ 11/4 & -1/2 & -3/4 \\ -1 & 0 & 0 \end{bmatrix} \qquad \because I = AA^{-1}$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 11/4 & -1/2 & -3/4 \\ -1 & 0 & 0 \end{bmatrix}$$

(j) Let $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

By using elementary row transformation, $A = IA$

i.e., $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ By $R_2 \to R_2 + 3R_1,\ R_3 \to R_3 - 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -7 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$ By $R_2 \to \frac{1}{9} R_2$

$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -7/9 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1/9 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$ By $R_1 \to R_1 - 3R_2, R_3 \to R_3 + 5R_2$

$\Rightarrow \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -7/9 \\ 0 & 0 & 1/9 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 0 \\ 1/3 & 1/9 & 0 \\ -1/3 & 5/9 & 1 \end{bmatrix} A$ By $R_3 \to 9R_3$

$\Rightarrow \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -7/9 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 0 \\ 1/3 & 1/9 & 0 \\ -3 & 5 & 9 \end{bmatrix} A$ By $R_1 \to R_1 - \frac{1}{3} R_3, R_2 \to R_2 + \frac{7}{9} R_3$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A$ $\because I = A^{-1}A$

Hence, $A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$

(k) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{pmatrix}$ $\because A = IA$ $\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$

By $R_2 \to R_2 - 2R_1$, $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 7 & 3 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$

By $R_2 \to R_2 - 3R_3$, $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} A$

By $R_1 \to R_1 + R_2$ & $R_3 \to R_3 - 2R_2$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -3 \\ -2 & 1 & -3 \\ 4 & -2 & 7 \end{pmatrix} A$

$\because A^{-1}A = I, \quad \therefore A^{-1} = \begin{pmatrix} -1 & 1 & -3 \\ -2 & 1 & -3 \\ 4 & -2 & 7 \end{pmatrix}$

(l) Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Using elementary row operations, A = I A we have,

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_1 \to \frac{1}{2}R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_2 \to R_2 - 5R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -5/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \text{By } R_3 \to R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -5/2 & 1 & 0 \\ 5/2 & -1 & 1 \end{bmatrix} A \qquad \text{By } R_3 \to 2R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -5/2 & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix} A \qquad \text{By } R_2 \to R_2 - \frac{5}{2}R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A \qquad \text{By } R_1 \to R_1 + \frac{1}{2}R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A \qquad \therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \quad \{\because I = A^{-1}A.$$

Q28. Let $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}, B = \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$

Now, $AB = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$

$$\Rightarrow AB = \begin{bmatrix} \cos^2\theta\cos^2\beta + \cos\theta\sin\theta\cos\beta\sin\beta & \cos^2\theta\cos\beta\sin\beta + \cos\theta\sin\theta\sin^2\beta \\ \cos\theta\sin\theta\cos^2\beta + \sin^2\theta\cos\beta\sin\beta & \cos\theta\sin\theta\cos\beta\sin\beta + \sin^2\theta\sin^2\beta \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} \cos\theta\cos\beta(\cos\theta\cos\beta + \sin\theta\sin\beta) & \sin\beta\cos\theta(\cos\theta\cos\beta + \sin\theta\sin\beta) \\ \sin\theta\cos\beta(\cos\theta\cos\beta + \sin\theta\sin\beta) & \sin\theta\sin\beta(\cos\theta\cos\beta + \sin\theta\sin\beta) \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} \cos\theta\cos\beta\cos(\theta-\beta) & \sin\beta\cos\theta\cos(\theta-\beta) \\ \sin\theta\cos\beta\cos(\theta-\beta) & \sin\theta\sin\beta\cos(\theta-\beta) \end{bmatrix}$$

If θ & β differ by an odd multiple of $\frac{\pi}{2}$ i.e., $(\theta-\beta)=(2n-1)\frac{\pi}{2}, n\in Z$ then $\cos(\theta-\beta)=0$

$$\Rightarrow AB = \begin{bmatrix} \cos\theta\cos\beta\times 0 & \sin\beta\cos\theta\times 0 \\ \sin\theta\cos\beta\times 0 & \sin\theta\sin\beta\times 0 \end{bmatrix} \qquad \therefore AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \quad \text{Hence Proved.}$$

Q29. LHS: $\left(\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$

$$\Rightarrow \quad = \begin{bmatrix} 1+\omega & \omega+\omega^2 & \omega^2+1 \\ \omega+\omega^2 & \omega^2+1 & 1+\omega \\ \omega^2+\omega & \omega^2+1 & \omega+1 \end{bmatrix}\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \qquad [\because 1+\omega+\omega^2=0$$

$$\Rightarrow \quad = \begin{bmatrix} -\omega^2 & -1 & -\omega \\ -1 & -\omega & -\omega^2 \\ -1 & -\omega & -\omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \qquad \Rightarrow \quad = \begin{bmatrix} -\omega^2-\omega-\omega^3 \\ -1-\omega^2-\omega^4 \\ -1-\omega^2-\omega^4 \end{bmatrix}$$

$$\Rightarrow \quad = \begin{bmatrix} -\omega(\omega+1+\omega^2) \\ -1-\omega^2-\omega^3\omega \\ -1-\omega^2-\omega^3\omega \end{bmatrix} \qquad \Rightarrow \quad = \begin{bmatrix} -\omega\times 0 \\ -(1+\omega^2+\omega) \\ -(1+\omega^2+\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \text{RHS}$$

Q30. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \qquad \Rightarrow A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$

As $AA^T = 9I_3$, $\therefore \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+2+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

By equality of matrices,

$a+4+2b=0 \qquad \Rightarrow a+2b=-4 \qquad$ — (I)

& $2a+2-2b=0 \qquad \Rightarrow a-b=-1 \qquad$ — (II)

Solving (I) & (II), we get : $a=-2, b=-1$

Q31. The given information is expressed in matrix :

School

		A	B
Appeared	→	25	35
Got through exam	→	20	20
Secured full marks	→	15	10

Q32. Given $A = \begin{bmatrix} 8 & 16 \\ 32 & 48 \end{bmatrix} \qquad \Rightarrow 7A = 7\begin{bmatrix} 8 & 16 \\ 32 & 48 \end{bmatrix} \qquad \therefore 7A = \begin{bmatrix} 56 & 112 \\ 224 & 336 \end{bmatrix}.$

It represents the no. of table fans and ceiling fans that the manufacturing units x and y produce in 7 days.

Q33. We have $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ $\therefore A^2 = A.A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{pmatrix}$

So, $A^2 - 5A + 4I = \begin{pmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{pmatrix} - 5\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow A^2 - 5A + 4I = \begin{pmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{pmatrix} - \begin{pmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$\therefore A^2 - 5A + 4I = \begin{pmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{pmatrix}$.

Also $A^2 - 5A + 4I + X = O$ $\Rightarrow \begin{pmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{pmatrix} + X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore X = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{pmatrix}$.